

Maximal PSL_2 subgroups of exceptional groups of Lie type

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Abstract

In this article we study embeddings of $\mathrm{PSL}_2(q_0)$ into exceptional groups $G(q)$ for $G = F_4, E_6, E_7$, and q_0 and q powers of the same prime p . With a few possible exceptions, we prove that there are no maximal subgroups with socle such a simple group inside an almost simple group with socle $G(q)$, except for those that arise as fixed points of a maximal positive-dimensional subgroup of the corresponding algebraic group.

In the few remaining cases we provide considerable information about a potential maximal subgroup.

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1 Introduction

Classifying the maximal subgroups of a finite group is one of the most fundamental problems in the field of finite group theory. Michael Aschbacher and Len Scott [5] reduced the problem for all finite groups to understanding $H^1(G, M)$ for all simple modules M for all finite simple groups G , and classifying all maximal subgroups of almost simple groups.

This paper is a contribution towards the latter, ambitious goal. For alternating and classical groups there is in some sense no complete answer, since the dimensions of the classical groups (and degrees of the alternating groups) tend to infinity, although there is substantial work in this direction. However, for sporadic and exceptional groups there is a possibility of a complete answer being known.

For sporadic groups, a complete answer is known for all groups but the Monster, and here we concentrate on exceptional groups of Lie type. There is a classification of maximal subgroups for exceptional groups $G = G(q)$ for G not of type F_4 , E_6 , 2E_6 , E_7 and E_8 already, and so we focus on these cases. What is known in the literature so far is summarized in Section 3, but broadly speaking, all maximal subgroups are known in these groups apart possibly from various almost simple maximal subgroups, and these are either a small list of simple groups that are not Lie type in defining characteristic, and if the potential maximal is Lie type in defining characteristic then what is left are groups of small rank and small field size, together with a large collection of possible subgroups $\mathrm{PSL}_2(p^a)$, the focus of this paper. We prove the following theorems, for any almost simple group of the appropriate type.

Theorem 1.1 Let p be a prime, $a \geq 1$ be an integer, and let q be a power of p . Let G be an almost simple group with socle $F_4(q)$, and suppose that H is an almost simple group with $F^*(H) = \mathrm{PSL}_2(p^a)$. If H is maximal in G then one of the following holds:

- (i) $p^a = 9$;
- (ii) $p^a = 13$, $H = \mathrm{PSL}_2(13)$ and is a Serre embedding;
- (iii) $q = p^a$, $p \geq 13$, $F^*(H) = \mathrm{PSL}_2(q)$, and H is the intersection of G with a maximal algebraic A_1 subgroup of the algebraic group F_4 .

The definition of a Serre embedding is given formally in Definition 4.7, but informally it is a copy of $\mathrm{PSL}_2(h+1)$ where h is the Coxeter number of G and this subgroup contains a regular unipotent element. (This subgroup is named after Serre as he constructed copies of $\mathrm{PSL}_2(h+1)$ (if $h+1$ is a prime) over all fields in [27].)

In recent work of Tim Burness and Donna Testerman, this case has been solved, and proved to be a subcase of (iii) above, so $p^a = 9$ is the only outstanding case. It seems difficult to remove the first possibility, although it might be possible using more advanced geometric techniques than employed here; of course no such maximal subgroup is known. In Section 8.2 we give more information about the case $p^a = 9$, where we give the action of a potential maximal subgroup on the minimal module; such a subgroup does exist, but is contained inside a positive-dimensional subgroup, and representation-theoretic techniques do not seem able to prove uniqueness.

Kay Magaard [24] proved Theorem 1.1 for $p \geq 5$ in his PhD thesis, with the extra condition that $q = 13$ in (ii).

For E_6 we have a complete theorem, as the Serre embedding can be shown to lie in F_4 .

Theorem 1.2 Let p be a prime, $a \geq 1$ be an integer, let q be a power of p , and let G be an almost simple group with socle either $E_6(q)$ or ${}^2E_6(q)$. There does not exist an almost simple maximal subgroup H of G with $F^*(H) = \mathrm{PSL}_2(p^a)$.

Almost all of this theorem was obtained by Aschbacher [4] using geometric techniques, where only the case $q = p^a = 11$ and H contains a semiregular unipotent element, from class $E_6(a_1)$, is left open; here we remove it using the Lie algebra structure of the adjoint module $L(G)$.

For E_7 , here we again have some potential exceptions, but not the case $p^a = 9$, which was completed in [9]. This time the difficult cases are the Serre embedding and $p^a = 7, 8, 25$.

Theorem 1.3 Let p be a prime, $a \geq 1$ be an integer and let q be a power of p . Let G be an almost simple group with socle $E_7(q)$, and suppose that H is an almost simple subgroup with $F^*(H) = \mathrm{PSL}_2(p^a)$. If H is maximal in G then one of the following holds:

- (i) $p^a = 7$, $p^a = 8$ or $p^a = 25$;
- (ii) $p^a = 19$, $H = \mathrm{PSL}_2(p^a)$ and is a Serre embedding;
- (iii) $q = p^a$, $p \geq 17$, $F^*(H) = \mathrm{PSL}_2(q)$, and H is the intersection of G with a maximal algebraic A_1 subgroup of the algebraic group E_7 .

Again, Burness and Testerman have showed that (ii) is a subcase of (iii). In the case (i) where $p^a = 8$, we can give the composition factors of H on the minimal module, and can give the precise module structure as well whenever $8 \mid q$. For $p^a = 7$, there are unresolved cases of potential copies of $\mathrm{PSL}_2(7)$ where the preimage of the subgroup in the simply connected version of E_7 is both $2 \times \mathrm{PSL}_2(7)$ and $\mathrm{SL}_2(7)$. In both cases the module structures on the minimal module can be given precisely, but it seems difficult to progress

further using these techniques. In the case of $p^a = 25$, this is a copy of $\mathrm{SL}_2(25)$ inside the simply connected version of E_7 with centres coinciding, and we have complete information about the module structures on both the minimal and adjoint modules. If it exists then it is a maximal subgroup of $E_7(q)$ for the smallest q into which the group embeds.

We do not deal with maximal subgroups of E_8 here, and only consider it for certain lemmas, which will be useful in a later treatment of this case. A rough estimate is that, with current techniques, attempting E_8 here would result in many unresolved cases and double the length of this work. For exceptional groups other than E_8 , the minimal module has dimension much smaller than the dimension of the group (as an algebraic group) and we can use representation theory to analyse this module. We can still do things with the Lie algebra for E_8 , as we did in [9],

The strategy for the proofs of these theorems is given in Section 7, and relies heavily on computer calculations in three ways:

- (i) The first is to compute the traces of semisimple elements of large order on various modules for exceptional groups. Tables of these traces are available for elements of small order, but we need them for very large orders, sometimes in the hundreds. For this we can use the program that Litterick produced in his PhD thesis [22], or construct the normalizer of a torus explicitly in Magma and take the conjugacy classes, then compute their eigenvalues. (Litterick has produced a much faster algorithm for computing traces of elements on fundamental modules, but we do not need this for our cases.)
- (ii) The second is to do large linear algebra problems. To find all sets of composition factors that could arise as the composition factors of the restriction of a G -module to a subgroup H involves checking many possible combinations against these large lists of possible sets of composition factors. This is done to reduce the possible module structures for the subgroup on the minimal and adjoint modules, and was also used in [22].
- (iii) The third is to construct explicit modules for finite groups, and show that certain module structures cannot exist. This would be possible by hand, at least in some cases, but incredibly complicated and prone to mistakes. In each case, a clear recipe is given for how to reproduce the module we construct to ease verifiability.

With these three uses of a computer in mind, the rest of the argument is done by hand, in Sections 8 to 12.

The structure of this article is as follows: in the next section we give notation and some preliminary results. In Section 3 we give information about maximal subgroups of finite and algebraic exceptional groups, and in the following section we give lots of information about unipotent and semisimple elements of exceptional groups, together with information about \mathfrak{sl}_2 -subalgebras of exceptional Lie algebras. Section 5 gives information about modules for $\mathrm{SL}_2(p^a)$, and the section after gives some constructions of PSL_2 s inside E_6 in characteristic 3.

We then launch into the proof proper, with Section 7 giving an outline of the strategy of the proof, Sections 8 and 9 proving the results for F_4 and E_6 , and then the three sections after doing E_7 in characteristic 2, and then E_7 in odd characteristic, split into two sections according as the embedding into the simply connected group is $2 \times \mathrm{PSL}_2(p^a)$ and $\mathrm{SL}_2(p^a)$.

The appendix gives some information about the composition factors of the reductive and parabolic maximal subgroups of F_4 , E_6 and E_7 on the minimal and adjoint modules, information that is well known but given here for ease of reference.

2 Notation and preliminaries

In this section we give the notation that we need, both for groups and for modules, and give a few preliminary results.

Throughout this paper, $G = G(k)$ will denote an exceptional finite group of Lie type defined over k , a field of characteristic $p \geq 2$. More specifically, let \bar{G} be a simple, simply connected algebraic group of exceptional type, equipped with a Frobenius endomorphism σ and set $G = \bar{G}^\sigma$. The precise types of G that we are interested in are those exceptional groups whose maximal subgroups are not yet known, i.e., $F_4(q)$, $E_6(q)$, ${}^2E_6(q)$, $E_7(q)$ and $E_8(q)$, although we do not do much in the case of $E_8(q)$, and often will exclude it from consideration.

Notice that we consider the simply connected version of G , so $E_7(k)$ possesses a centre when p is odd. We want the simply connected versions in order to work with the minimal module and the adjoint module simultaneously. Where this is particularly important we will remind the reader, for example when considering $\mathrm{PSL}_2(p^a)$ embedded in the simple group of type E_7 , where in $E_7(k)$ we can embed either $\mathrm{SL}_2(p^a)$ in $E_7(k)$ with the centres coinciding or $2 \times \mathrm{PSL}_2(p^a)$ into $E_7(k)$ with the centres coinciding, representing the two possible preimages of a copy of $\mathrm{PSL}_2(p^a)$ in the simple group. If G possesses a graph automorphism of order 2, denote this by τ ; we will remind the reader of this notation when we use it.

We let \bar{G} be an almost simple group with socle $G/Z(G)$. The maximal subgroups M of \bar{G} split into three categories: $M \cap G$ is a maximal subgroup of G , $M \cap G$ is not a maximal subgroup of G , and $G \leq M$. The third collection are easily computed, and the first can be deduced from a list of maximal subgroups of G . However, the second, called *novelty* maximal subgroups, cannot easily be seen from the maximal subgroups of G . They arise in the following manner: let H be a subgroup that is not maximal in a simple group X , but H is normalized by a group of outer automorphisms A of X while every proper subgroup of X properly containing H is not normalized by it. In this case, HA is a maximal subgroup of XA . However, it is of course very difficult to understand these if one is simply given a list of maximal subgroups of X , so we will prove more than simply that a given subgroup is not maximal in the simple group, but that it is contained in stabilizers of various subspaces of a given module, enough that we can see that it does not form a novelty maximal subgroup.

The modules that we normally consider are the two smallest non-trivial ones. Write V_{\min} for one of the minimal modules for G , namely $L(\lambda_4)$ for F_4 , either $L(\lambda_1)$ or $L(\lambda_6)$ for E_6 and 2E_6 , $L(\lambda_7)$ for E_7 and not defined for E_8 . We write $L(G)$ for the Lie algebra or adjoint module, which is $L(\lambda_1)$, $L(\lambda_2)$, $L(\lambda_1)$ and $L(\lambda_1)$ respectively. If $L(G)$ has a trivial composition factor so is not irreducible, which occurs in E_7 in characteristic 2 and E_6 in characteristic 3, let $L(G)'$ denote the other simple factor, and in other cases let $L(G)' = L(G)$. These two modules have the following dimensions:

Group	$\dim(V_{\min})$	$\dim(L(G)')$
F_4	$26 - \delta_{p,3}$	52
E_6	27	$78 - \delta_{p,3}$
E_7	56	$133 - \delta_{p,2}$
E_8	248	248

If $G = F_4$ in characteristic 2, $L(G)$ has factors V_{\min} and V_{\min}^τ , where τ denotes the graph automorphism of G , so in this case we will not consider $L(G)$ at all but these two modules. In all other cases, $L(G)'$ is irreducible.

We now introduce some notation for modules. All modules will be finite dimensional and will normally

be defined over k , the field over which G is defined. If H is a group, let $\text{Irr}(H)$ denote the set of irreducible modules over the field, which is always k unless otherwise stated. As usual write \oplus and \otimes for the direct sum and tensor product of two modules. Let Λ^i and S^i denote the exterior and symmetric powers. We write $M \downarrow_H$ for the restriction of M to H , and write $\text{soc}^i(M)$ for the i th socle layer and $\text{rad}^i(M)$ for the i th radical layer of M . Write $\text{top}(M)$ for the top of M , i.e., $M/\text{rad}(M)$, and $\text{cf}(M)$ for the composition factors of M as a multiset. Let $H^1(H, M)$ denote the 1-cohomology group of M , and in general $\text{Ext}^1(M, M')$ denote the group of extensions with submodule M' and quotient M . The projective cover of a module M will be denoted by $P(M)$.

We will often have to talk about the structures of modules, as in their socle layers. If M is a module with socle A and second socle B then we can write

$$\begin{array}{c} B \\ A \end{array}$$

for this structure; however, this is often too space-consuming when we have many socle layers, and so we also write B/A for this module.

We also introduce the concepts of radical and residual. If I is a subset of $\text{Irr}(H)$, then the I -radical of M is the largest submodule of M whose composition factors lie in I , and let $I' = \text{Irr}(H) \setminus I$. The I -residual of M is the smallest submodule whose quotient has composition factors in I .

One lemma that we occasionally use, that can be quite powerful, relates the minimal and adjoint modules for exceptional groups. We place it here because there seems no more appropriate place.

Lemma 2.1 Let G be one of F_4 , E_6 and E_7 .

- (i) Let $G = F_4$. If $p = 3$ then $L(G)$ is a submodule of $\Lambda^2(V_{\min})$. If $p \geq 5$ then $L(G)$ is a summand of $\Lambda^2(V_{\min})$.
- (ii) Let $G = E_6$. If $p = 2$ then $L(G)$ is a submodule of $V_{\min} \otimes V_{\min}^*$. If $p = 3$ then the socle of $V_{\min} \otimes V_{\min}^*$ is 1-dimensional, and quotienting out by this, $L(G)'$ is a submodule. If $p \geq 5$ then $L(G)$ is a summand of $V_{\min} \otimes V_{\min}^*$.
- (iii) Let $G = E_7$. If $p = 2$ then the socle of $\Lambda^2(V_{\min})$ is 1-dimensional, and quotienting out by this, $L(G)'$ is a submodule. If $p = 3$ then $L(G)$ is a submodule of $S^2(V_{\min})$. If $p \geq 5$ then $L(G)$ is a summand of $S^2(V_{\min})$.

In many cases we want to prove that a module has a particular composition factor as a submodule or quotient, often the trivial module. Thus we need a method of proving that a particular composition factor is always a submodule or quotient in any module with those factors. This is the idea of pressure.

Suppose that H is a finite group such that $O^p(H) = H$, and such that for all simple modules M over a field k , $H^1(H, M) = H^1(H, M^*)$. The *pressure* of a module V for H is the quantity

$$\sum_{M \in \text{cf}(V)} \dim H^1(H, M) - \delta_{M,k}.$$

Results on pressure have occurred in the literature before, with the most general being in [9]. Another generalization of this allows us to understand the situation of forcing a module from a collection \mathcal{M} of simple modules to be a submodule of a given module V . If \mathcal{M} is a collection of simple modules for a group

H , with $\text{Ext}^1(M, M') = 0$ for all $M, M' \in \mathcal{M}$, and such that $\text{Ext}^1(A, M) = \text{Ext}^1(M, A)$ for all simple modules A and M with $M \in \mathcal{M}$, then the \mathcal{M} -pressure of a module V is the quantity

$$\sum_{M' \in \text{cf}(V)} \sum_{M \in \mathcal{M}} \text{Ext}^1(M, M') - \delta_{M, M'}.$$

The lemma from [9] directly generalizes to \mathcal{M} -pressure, with the exact same proof, and we give it now.

Lemma 2.2 Suppose that H is a finite group, and let \mathcal{M} be a set of simple modules for H such that $\text{Ext}^1(M, M') = 0$ for all $M, M' \in \mathcal{M}$, and $\text{Ext}^1(M, A) = \text{Ext}^1(A, M)$ for all $M \in \mathcal{M}$ and all simple modules A . Let V be a module for H of \mathcal{M} -pressure n .

- (i) If $n < 0$ then $\text{Hom}(M, V) \neq 0$ for some $M \in \mathcal{M}$, i.e., V has a simple submodule isomorphic to some $M \in \mathcal{M}$. If $n = 0$ then either $\text{Hom}(M, V) \neq 0$ or $\text{Hom}(V, M) \neq 0$, i.e., V has either a simple submodule or quotient isomorphic to some member of \mathcal{M} .
- (ii) More generally, if a composition factor of V has \mathcal{M} -pressure greater than n , then either $\text{Hom}(M, V) \neq 0$ or $\text{Hom}(V, M) \neq 0$ for some $M \in \mathcal{M}$.
- (iii) If $\text{Hom}(M, V) = \text{Hom}(V, M) = 0$ for all $M \in \mathcal{M}$, then any subquotient W of V has \mathcal{M} -pressure between $-n$ and n .

The concept of pressure can be used to prove that either V_{\min} or $L(G)$ possesses a trivial submodule or quotient when restricted to H . We therefore would like to know whether that is enough in some circumstances to conclude that H is contained within a σ -stable, positive-dimensional subgroup of G .

Lemma 2.3 [[9, Lemma 1.4]] Let $G = \mathbb{G}^\sigma$ be one of F_4 , E_6 , 2E_6 , E_7 or E_8 . Let $H \leq G^\sigma$, acting on a module V defined over k , where V is either V_{\min} or $L(G)'$ and k is the underlying field of G . If one of the following holds, then H is contained in a σ -stable, positive-dimensional subgroup of G :

- (i) H fixes a 1-space or hyperplane of V_{\min} or $L(G)$;
- (ii) $G = F_4$, E_6 , 2E_6 or E_7 , and H fixes a 2-space or a space of codimension 2 in V_{\min} ;
- (iii) $G = E_6$ or 2E_6 , and H fixes a 3-space or a 24-space of V_{\min} .

In the next section we consider the set of maximal positive-dimensional subgroups, and this lemma will more or less translate across to the almost simple group \tilde{G} .

We end with giving the line stabilizers for the minimal modules for the finite groups $E_6(k)$ and $E_7(k)$. These have appeared in the literature before, and we take these from [15, Lemmas 5.4 and 4.3].

Lemma 2.4 Let $G = E_6(q)$. There are three orbits of lines of the action of G on V_{\min} , with line stabilizers as follows:

- (i) $F_4(q)$ acting on V_{\min} as $L(\lambda_4) \oplus L(0)$;
- (ii) a D_5 -parabolic subgroup; $q^{16}D_5(q).(q-1)$, acting uniserially as $L(\lambda_1)/L(\lambda_4)/L(0)$.
- (iii) a subgroup $q^{16}.B_4(q).(q-1)$ acting indecomposably as $L(0), L(\lambda_1)/L(\lambda_4)/L(0)$.

Lemma 2.5 Let $G = E_7(q)$. There are five orbits of lines of the action of G on V_{\min} , with line stabilizers as follows:

- (i) $E_6(q).2$ (the graph automorphism) acting semisimply with composition factors of dimensions 54, 1, 1;
- (ii) ${}^2E_6(q).2$ (the graph automorphism) acting semisimply with composition factors of dimensions 54, 1, 1;
- (iii) an E_6 -parabolic subgroup $q^{27}.E_6(q).(q-1)$ acting uniserially as $L(0)/L(\lambda_1)/L(\lambda_6)/L(0)$;
- (iv) a subgroup $q^{1+32}.B_5(q).(q-1)$ acting uniserially as $L(0)/L(\lambda_1)/L(\lambda_5)/L(\lambda_1)/L(0)$;
- (v) a subgroup $q^{26}.F_4(q).(q-1)$ acting indecomposably as $L(0), L(0)/L(\lambda_4)/L(\lambda_4)/L(0), L(0)$.

3 Maximal subgroups

This section summarizes what is known about the maximal subgroups of the finite groups G and \bar{G} , and also the algebraic group \mathbb{G} , about which complete information on positive-dimensional maximal subgroups is known.

The maximal subgroups of positive dimension in \mathbb{G} are given in [20], and given \mathbb{G} we denote by \mathcal{X} this collection; write \mathcal{X}^σ for the fixed-point sets \mathbf{X}^σ for $\mathbf{X} \in \mathcal{X}$ being σ -stable. Note that we also include in \mathcal{X}^σ the fixed points of G under a field, graph, or field-graph automorphism of prime order (so, for example, ${}^2E_6(p^2)$ and $E_6(p)$ inside $E_6(p^2)$). If \bar{G} is almost simple rather than merely simple, the set \mathcal{X}^σ shall be taken to mean the normalizers in \bar{G} of the elements of \mathcal{X}^σ for $F^*(\bar{G})$.

While the maximal subgroups of \mathbb{G} are known, the maximal subgroups of G and \bar{G} are of course not. We start with a broad characterization of the maximal subgroups of \bar{G} , given in [7] and [17, Theorem 2].

Theorem 3.1 Let M be a maximal subgroup of \bar{G} not containing $F^*(\bar{G})$. One of the following holds:

- (i) M is a member of \mathcal{X}^σ ;
- (ii) M is the normalizer of an elementary abelian r -group for some $r \neq p$;
- (iii) $M = (\text{Alt}(5) \times \text{Sym}(6)) \cdot 2$ and $G = E_8$ with $p > 5$;
- (iv) M is almost simple.

The algebraic groups containing the subgroups in (i) are known and are the fixed points of those in [20]; the subgroups in (ii) are known and given in [8]; the subgroup (iii) was discovered by Borovik and is unique up to conjugacy. The potential subgroups in (iv) have been steadily reduced over the last two decades. We start with those almost simple groups that are not Lie type in defining characteristic. Here the list is fairly short and given in [19], but note that a fair number of these have been eliminated in a variety of papers, too numerous to list here, but we mention the papers [23] and [9] for all Lie type groups, and with F_4 and E_6 having almost all possibilities for M removed by Magaard and Aschbacher in [24] and [4] respectively. The author has also made progress on eliminating still more of this list and proving uniqueness of various maximal subgroups, with details appearing elsewhere.

For M a group of Lie type in defining characteristic, define $t(G)$ and $v(G)$ to be the following integers:

$$t(G_2) = 12, \quad t(F_4) = 68, \quad t(E_6) = 124, \quad t(E_7) = 388, \quad t(E_8) = 1312.$$

$$v(G_2) = 4, \quad v(F_4) = 18, \quad v(E_6) = 18, \quad v(E_7) = 75, \quad v(E_8) = 1312.$$

The rank of M is at most half the rank of G by [16] and [21]. Furthermore, for those groups we have the following possibilities by [18]:

- (i) $M(q)$ has semisimple rank at most half that of G , $q \leq 9$, and $M(q)$ is not one of $\text{PSL}_2(q)$, ${}^2B_2(q)$ and ${}^2B_2(q)$;
- (ii) $\text{PSL}_3(16)$ and $\text{PSU}_3(16)$;
- (iii) $\text{PSL}_2(q)$, ${}^2B_2(q)$ and ${}^2G_2(q)$ for $q \leq \gcd(2, q-1) \cdot t(G)$.

The paper [10] allows us to replace $t(G)$ by $v(G)$ in (iii). For (ii), note that $\mathrm{PSL}_3(16)$ has elements of order 9, and $\mathrm{PSU}_3(16)$ has elements of order 255, so neither case can occur for $G \neq E_8$ by [10, Theorem 1.1]. The author, Kay Magaard and Chris Parker have removed almost all of (i) for $G \neq E_8$ in work in preparation, as well as the Suzuki and Ree groups from (iii), leaving just $\mathrm{PSL}_2(p^a)$, which we consider in this paper.

We can therefore assume, from now on, that $p^a \leq \gcd(2, p-1) \cdot v(G)$. We remind the reader at the start of each section the value of $v(G)$.

We wish to end this section with a result that states that if H is a copy of $\mathrm{PSL}_2(p^a)$ inside an exceptional group of Lie type (other than E_8), then $N_{\bar{G}}(H)$ is either an almost simple maximal subgroup or is inside a member of \mathcal{X}^σ . The rest of this paper will be spent proving that the latter case holds rather than the former, but for this section we will need to have some exceptions. One source of possible exceptions is that $N_{\bar{G}}(H)$ is contained inside another maximal subgroup of \bar{G} other than those in \mathcal{X}^σ , for example a copy of $\mathrm{PSp}_6(p)$, which contains $\mathrm{PSL}_2(p^3)$.

The statement of the next result, and the proofs of the next two results, use ideas, definitions and techniques that will be introduced throughout this paper, but logically the results should be in this section. As such, the author recommends that the reader does not read the proofs of these results until after they have read the next few sections.

In order to reduce the list of exceptions that arise, we will remove some of the Lie type groups of medium rank appearing in (i) above. We start with a table giving the largest possible order of semisimple elements of various groups of Lie type. The group type appears on the left and the field size on the top. In each entry, there is a number which is the largest order of a semisimple element in the simple group, and if this is even we include the largest element of odd order in brackets, then for groups for which not all semisimple elements are real (type A only) we place after that the largest order of a real semisimple element.

Group	2	3	4	5	7	8	9
PSL_3	7, 3	13, 4	7, 5	31, 6	19, 8	73, 9	91, 10
PSL_4	15, 5	20 (13), 10	85, 17	39, 13	200 (171), 50	585, 65	205, 41
PSU_3	-	8 (7), 4	15, 5	8 (7), 6	48 (43), 8	21, 9	80 (73), 10
PSU_4	9, 5	8 (7), 8	65, 17	63, 26	86 (75), 25	513, 65	365, 82
PSp_4	5	5	17	13	25	65	41
PSp_6	15	20 (13)	85	78 (63)	200 (171)	585	410 (365)
$\mathrm{P}\Omega_7$	15	20 (13)	85	78 (63)	200 (171)	585	410 (365)
G_2	7	13	21	31	57	73	91

(We omit $\mathrm{PSU}_3(2)$, which is not simple, and consider the derived subgroup when the group is not simple, i.e., $\mathrm{PSp}_4(2)'$ and $G_2(2)'$.)

We now compare these numbers to $v(G)$: for F_4 if the first number is greater than 18 then the subgroup is a blueprint for V_{\min} ; for E_6 if the number in brackets is greater than 75 or the second number is greater than 18 then the subgroup is a blueprint for V_{\min} ; for E_7 the number in brackets needs to be greater than 75 for this. For example, subgroups isomorphic with $G_2(9)$ and $\mathrm{PSp}_6(8)$ are always blueprints for V_{\min} when inside F_4 , E_6 and E_7 , whereas $\mathrm{PSU}_4(7)$ is always a blueprint for V_{\min} for F_4 and E_6 , but not necessarily for E_7 .

We prove an intermediate proposition that will help in our stated goal of producing the result we mentioned about $N_{\bar{G}}(H)$ being either almost simple or in a member of \mathcal{X}^σ when H is $\mathrm{PSL}_2(p^a)$.

Proposition 3.2 Let G be one of F_4 , E_6 and E_7 .

- (i) For $p = 5, 7$, any copy of $H = \mathrm{PSp}_4(p)$ in G , or $H = \mathrm{Sp}_4(p)$ in $G = E_7(k)$ with $Z(H) = Z(G)$, is a blueprint for V_{\min} .
- (ii) For $p = 5, 7$, any copy of $H = \mathrm{PSL}_4(p)$ or $\mathrm{PSU}_4(p)$ in G , or $H = 2 \cdot \mathrm{PSL}_4(p)$ or $H = 2 \cdot \mathrm{PSU}_4(p)$ in $G = E_7(k)$ with $Z(H) = Z(G)$, is a blueprint for V_{\min} .
- (iii) Let p be an odd prime and $a \geq 1$. Any copy of $H = \mathrm{PSp}_6(p^a)$ in G , or $H = \mathrm{Sp}_6(p^a)$ in $G = E_7(k)$ with $Z(H) = Z(G)$, is a blueprint for V_{\min} .
- (iv) Let p be an odd prime and $a \geq 1$. Any copy of $H = \Omega_7(p^a)$ in G , or $H = \mathrm{Spin}_7(p^a)$ in $G = E_7(k)$ with $Z(H) = Z(G)$, is a blueprint for V_{\min} .

Proof: We prove (i) and (ii) for $p = 5$ first. For the first part, we first compute the conspicuous set of composition factors for $V_{\min} \downarrow_H$ in the case of $G = E_7$. The simple modules of dimension at most 56 are 1, 5, 10, 13, 30, 35_1 , 35_2 and 55. The only conspicuous set of composition factors is $10^2, 5^6, 1^6$, and since these composition factors have no extensions with each other, $V_{\min} \downarrow_H$ is semisimple, thus a unipotent element u from the conjugacy class of H with the largest centralizer acts on V_{\min} as $3^2, 2^{16}, 1^{18}$, a generic class. This proves the result since generic classes are blueprints for V_{\min} . Of course, since the minimal modules for F_4 and E_7 are submodules of V_{\min} , the result holds for F_4 and E_6 as well.

If $H = \mathrm{Sp}_4(5) \leq E_7(k)$ with centres coinciding, then the involutions in H act on faithful modules with trace 0, not allowed since the trace of an involution in E_7 is ± 8 . This proves (i).

As $\mathrm{SL}_4(5)$ and $\mathrm{SU}_4(5)$ contain $\mathrm{Sp}_4(5)$, and the centres of $\mathrm{SL}_4(5)$ and $\mathrm{SU}_4(5)$ contain the centre of $\mathrm{Sp}_4(5)$, we have that $\mathrm{PSP}_4(5) \leq \mathrm{PSL}_4(5), \mathrm{PSU}_4(5)$, and therefore (ii) holds as subgroups that contain blueprints are themselves blueprints.

For $p = 7$ the exact same proof holds, except that the dimensions of the simple modules are now 1, 5, 10, 14, 25, 35_1 , 35_2 and 54.

We now prove (iii). For $p^a \neq 3, 5$, the largest semisimple element of odd order has order greater than 75, so these are already blueprints for V_{\min} . If $H = \mathrm{PSp}_6(3)$ then there are only three simple modules of dimension at most 56, with dimensions 1, 14 and 21. The traces of elements of orders 5 and 7 are enough to prove that H does not embed in $G = E_7$, and hence not in its subgroups. If $H = \mathrm{Sp}_6(3)$ then the appropriate simple modules have dimensions 6, 14 and 50, and traces of elements of order 5 are enough to prove that the only conspicuous set of composition factors for $V_{\min} \downarrow_H$ is $14, 6^7$. There are no extensions between composition factors, so this is semisimple, and the action of a unipotent element with largest centralizer in H is $2^{12}, 1^{32}$, a generic class. We conclude that H is a blueprint for V_{\min} and therefore so is any subgroup containing H , as needed.

For $p = 5$ all of the same statements hold except we only need traces of elements of order 3 to prove that $\mathrm{PSp}_6(5)$ does not embed, and for $\mathrm{Sp}_6(5)$ elements of orders 2 and 3 suffice.

Finally, we consider (iv). Since the semisimple elements have the same orders in $\Omega_7(p^a)$ as $\mathrm{PSp}_6(p^a)$, we again need only consider $p^a = 3, 5$. For $p^a = 3$, the simple modules for $H = \Omega_7(3)$ of dimension at most 56 are 1, 7, 27 and 35. The traces of elements of orders 2 and 4 are enough to find the unique conspicuous set of composition factors, $21^2, 7^2$. and since there are no extensions between these modules $V_{\min} \downarrow_H$ is semisimple. A unipotent element H with maximal centralizer size acts on this module with blocks $3^2, 2^{16}, 1^{18}$, which is generic by [13, Table 7], so that H is a blueprint for V_{\min} . In the other case of $H = \mathrm{Spin}_7(3)$, a non-central involution in H has trace 0 on all faithful modules, and so since an involution in E_7 has trace ± 8 , we cannot get this case.

The exact same proof works for $p = 5$ except we use traces of elements of orders 2 and 3 to eliminate all but one set of composition factors. \square

From this we can see that if H is a potential maximal subgroup, and we prove that H is contained inside a larger subgroup that is not G , then for almost all possibilities for H , either H must lie inside another almost simple group or it lies inside a member of \mathcal{X}^σ . This is made formal with the following proposition.

Proposition 3.3 Let $H = \text{PSL}_2(p^a)$, let $G = G(k)$ be an exceptional group of Lie type in characteristic p other than E_8 , and let \bar{G} be an almost simple group with socle G . If $H \leq G$ then one of the following holds:

- (i) $N_{\bar{G}}(H)$ is contained in a member of \mathcal{X}^σ ;
- (ii) $N_{\bar{G}}(H)$ is an almost simple maximal subgroup of \bar{G} with socle H ;
- (iii) one of the following holds:
 - (a) $G = G_2$, $p^a = 4, 7$;
 - (b) $G = F_4$, $p^a = 4, 5, 7, 8, 9, 11$;
 - (c) $G = E_6$, $p^a = 4, 5, 7, 8, 9, 11, 16, 25$ (for $p^a = 25$, $H \leq {}^2F_4(2)'$);
 - (d) $G = E_7$, $p^a = 4, 5, 7, 8, 9, 11, 16, 25, 64$ (for $p^a = 25$, $H \leq Ru$).

Proof: As we wrote above, the classification-so-far of maximal subgroups of G (not equal to E_8) states that if M is a maximal subgroup then one of the following holds:

- (1) M is a member of \mathcal{X}^σ ;
- (2) M is an exotic local subgroup;
- (3) $F^*(M)$ is $\text{PSL}_2(p^a)$ for $p^a \leq \gcd(2, p-1) \cdot v(G)$;
- (4) $F^*(M)$ is ${}^2B_2(p^a)$ for $p^a \leq 32$ or ${}^2G_2(p^a)$ for $p^a \leq 27$;
- (5) $F^*(M)$ is a simple group of Lie type in characteristic p , whose untwisted rank is at most half of that of G , and whose field of definition is at most 9;
- (6) $F^*(M)$ is a simple group not of Lie type in defining characteristic, and is one of the groups in [19, Section 10].

If we assume, in the proposition, that neither (i) nor (ii) holds, then there must be a maximal subgroup, M , containing H , and such that M arises in (2), (3), (4), (5) or (6). The Suzuki and small Ree groups cannot contain H and so we can exclude (4), and (3) is dealt with as we get (i). For (5) we have the restrictions imposed above, and from (6) we can exclude all alternating groups other than $\text{Alt}(6)$ and $\text{Alt}(7)$ by [9].

The exotic local subgroups for $G \neq E_8$ have composition factors either cyclic groups, or $\text{SL}_3(2)$ (G_2 and above), $\text{SL}_3(3)$ (F_4 and above) and $\text{SL}_3(5)$ (E_6 and above); the first two are minimal simple groups anyway, and the third contains only $\text{SL}_2(4)$, so we get $p^a = 4$ for E_6 , which is also contained in $\text{Alt}(6)$, for example, so we can exclude (2).

For $G = G_2$, the possible M that are not minimal simple are $\text{PSU}_3(3)$ which contains $\text{PSL}_2(7)$, J_1 in characteristic 11, which contains $\text{PSL}_2(11)$, and J_2 in characteristic 2, which contains $\text{SL}_2(4)$, which completes the proof for this case. (We exclude $p^a = 11 \geq 2 \cdot v(G_2)$.)

For F_4 , $\text{Alt}(6)$ contains $\text{PSL}_2(5) = \text{SL}_2(4)$, and from Lie type in defining characteristic we can place $\text{PSL}_2(p^a)$ inside another Lie type group in characteristic p , say $\text{PSL}_3(p^a)$, for $p^a \leq 9$. All other possibilities for M cannot include $\text{PSL}_2(p^a)$ for $p^a \neq 4, 5, 7, 8, 9, 11$.

For E_6 and E_7 , from (5) we get $\text{Sp}_4(8)$, which contains $\text{SL}_2(64)$ (but this fails $v(E_6)$), and $\text{Sp}_4(4)$ which contains $\text{SL}_2(16)$. From (6), note that $\text{PSL}_2(25)$ lies inside ${}^2F_4(2)'$ and the sporadic group Ru . The former of these must act irreducibly on the minimal module for E_6 , and with factors $27^2, 1^2$ on the minimal module for E_7 , thus lies inside a E_6 -parabolic and is indeed a blueprint for V_{\min} in this case (see the table in [23]). The latter only embeds in E_7 , and as $28 \oplus 28^*$. \square

4 Unipotent and semisimple elements

This section collects together a variety of facts about unipotent and semisimple elements in groups of Lie type. We consider criteria for semisimple elements that are blueprints (see Definition 4.5 below), summarizing results of [10] and providing one more example of the calculations performed there. We then move on to considering modules for SL_2 , and how the weight spaces of the module and the eigenvalues of elements of SL_2 interact, with the aim of finding blueprint elements and subgroups of SL_2 .

4.1 Actions of unipotent elements

Let G be a simple algebraic group in characteristic p . The Bala–Carter–Pommerening labelling system for the unipotent classes, as used in a slightly modified form (to deal with interpolation of extra classes in certain bad characteristics) in our main reference [13] for unipotent classes of exceptional groups, gives us a way to discuss unipotent classes that is independent of the characteristic p of G . We may therefore compare the action of a unipotent class on a fixed simple module for different primes.

As is well known, any matrix of order a power of a prime p defined over a field of characteristic p can be written in Jordan normal form, with the conjugacy class in the general linear group being determined by the sizes of the Jordan blocks. Thus, if u is a unipotent element of an algebraic group G then for every module for G of dimension n we can associate a partition of n , the sizes of the various Jordan blocks in the action of u on the module. We use the notation for this, and unipotent classes, from [13], which determines the Jordan block structure of the action of all unipotent classes of exceptional groups on the minimal and adjoint modules.

The only cases we will need that are not covered in [13] are when the minimal or adjoint module is not simple, e.g., F_4 in characteristic 3. The next lemma gives the actions of the unipotent classes on the 25-dimensional simple module, on the 26-dimensional module 25/1, which is the ‘minimal module’ for other characteristics, and on the 27-dimensional minimal module for E_6 , which has structure 1/25/1.

Lemma 4.1 Let u be a unipotent element in $F_4(3^n)$. The Jordan blocks of the action of u on the 25-dimensional minimal module, together with the extension 25/1 and the minimal module for E_6 is one of those given in Table 4.1.

Proof: The actions of the unipotent elements on the 26-dimensional module are given in [13, Table 3], and using a computer, a representative of each of the classes was constructed in $F_4(3)$. The actions on the 25-dimensional composition factors were then computed, and are as above. The classes on the 25/1 are exactly those in [13, Table 3], and the corresponding classes for E_6 are in [13, Table 5]. \square

Using a computer and constructing classes manually is the method by which we prove the next two lemmas, which we include for completeness.

Lemma 4.2 Let u be a unipotent element in $E_6(3^n)$. The Jordan blocks of the action of u on the 77-dimensional Lie algebra module $L(G)'$ are obtained from the action on $L(G)$ by removing a Jordan block of size 1, except in the cases listed in Table 4.2.

Lemma 4.3 Let u be a unipotent element in $E_7(2^n)$. The Jordan blocks of the action of u on the 132-dimensional Lie algebra module $L(G)'$ are obtained from the action on $L(G)$ by removing a Jordan block of size 1, for every unipotent class.

Class in F_4	Action on 25	Action on 25/1	Action on 1/25/1
A_1	$2^6, 1^{13}$	$2^6, 1^{14}$	$2^6, 1^{15}$
\tilde{A}_1	$3, 2^8, 1^6$	$3, 2^8, 1^7$	$3, 2^8, 1^8$
$A_1 + \tilde{A}_1$	$3^3, 2^6, 1^4$	$3^3, 2^6, 1^5$	$3^3, 2^6, 1^6$
A_2	$3^6, 1^7$	$3^6, 1^8$	$3^6, 1^9$
$A_2 + \tilde{A}_1$	$3^7, 2^2$	$3^7, 2^2, 1$	$3^7, 2^2, 1^2$
\tilde{A}_2 and $\tilde{A}_2 + A_1$	$3^8, 1$	$3^8, 2$	3^9
B_2	$5, 4^4, 1^4$	$5, 4^4, 1^5$	$5, 4^4, 1^6$
$C_3(a_1)$	$5^2, 4^2, 3, 2^2$	$5^2, 4^2, 3, 2^2, 1$	$5^2, 4^2, 3, 2^2, 1^2$
$F_4(a_3)$	$5^3, 3^3, 1$	$5^3, 3^3, 1^2$	$5^3, 3^3, 1^3$
B_3	$7^3, 1^4$	$7^3, 1^5$	$7^3, 1^6$
$C_3, F_4(a_2)$	$9, 6^2, 3, 1$	$9, 6^2, 3, 2$	$9, 6^2, 3^2$
$F_4(a_1)$	$9^2, 7$	$9^2, 7, 1$	$9^2, 7, 1^2$
F_4	$15, 9, 1$	$15, 9, 2$	$15, 9, 3$

Table 4.1: Actions of unipotent elements on V_{\min} and its extensions for F_4 in characteristic 3

Class in E_6	Action on $L(G)'$	Action on $L(G)$
$2A_2$	$3^{23}, 1^8$	$3^{23}, 2, 1^7$
$2A_2 + A_1$	$3^{24}, 2^2, 1$	$3^{24}, 2^3$
A_5	$9^3, 8^2, 6^4, 3^2, 1^4$	$9^3, 8^2, 6^4, 3^2, 2, 1^3$
$E_6(a_3)$	$9^4, 7, 6^4, 3^3, 1$	$9^4, 7, 6^4, 3^3, 2$
$E_6(a_1)$	$9^8, 5$	$9^8, 6$
E_6	$19, 15^2, 9^3, 1$	$19, 15^2, 9^3, 2$

Table 4.2: Actions of unipotent elements on $L(G)'$ and $L(G)$ for E_6 in characteristic 3, where one does not obtain the former from the latter by removing a trivial Jordan block

We can see from the tables in [13] that for every unipotent class there is a set of primes P such that, for any prime $p \notin P$ the partition describing the Jordan block structure is the same.

Definition 4.4 Let G be an algebraic group and let M be a highest weight module for G . Let u be a unipotent element of G . If the Jordan block structure of u on M is the same as for cofinitely many primes, then u is said to be *generic* on M .

Thus, informally, the non-generic classes are those where the prime is in the set P described above, where the partition differs from the ‘usual’ one.

The reason that generic unipotent classes are interesting is that we can find ‘nice’ A_1 subgroups containing them, at least if the class has order p . To pin down the concept of ‘nice’, we introduce the following definition.

Definition 4.5 Let G be an algebraic group and let M be a module for G . A subgroup H of G is a *blueprint* for M if there exists a positive-dimensional subgroup X of G such that X and H stabilize the same subspaces of M . An element x is a blueprint for M if $\langle x \rangle$ is.

We may now state the lemma from [9] that defines ‘nice’.

Lemma 4.6 ([9, Lemma 1.2]) Suppose that $G = F_4, E_6, E_7$, and if $G = F_4$ then p is odd. If H is a finite subgroup of \tilde{G} and H contains a non-trivial unipotent element whose action on V_{\min} , $V_{\min} \oplus V_{\min}^*$ if $G = E_6$, or $L(G)$ is generic, then u and therefore H are blueprints for V_{\min} , $V_{\min} \oplus V_{\min}^*$, or $L(G)$ respectively, and in particular H is contained in a member of \mathcal{X}^σ .

Thus, if any subgroup H of an exceptional algebraic group G contains a unipotent element of order p that is generic for either the minimal or adjoint module, then H is contained inside an element of \mathcal{X} and indeed \mathcal{X}^σ if H is inside $G = G^\sigma$.

For large primes, we will often prove that H stabilizes a unique 3-dimensional submodule of $L(G)$, which must be a subalgebra of the Lie algebra. If this 3-dimensional submodule of $L(G) \downarrow_H$ is a summand then we may apply Proposition 4.17, but if the 3-dimensional submodule is not a summand then we cannot easily prove that it is an \mathfrak{sl}_2 , as it need not be simple. There is one case in particular where this occurs, which we refer to as a Serre embedding. These are embeddings of $\mathrm{PSL}_2(h+1)$ into an algebraic group, where h is the Coxeter number of the group.

Definition 4.7 Let G be an exceptional algebraic group with Coxeter number h , and let $p = h + 1$. A subgroup $H = \mathrm{PSL}_2(p)$ is a *Serre embedding* if the following conditions hold:

- (i) on $L(G)$, H stabilizes a unique 3-dimensional subspace;
- (ii) H contains a regular unipotent element.

In Section 4.6 we discuss subalgebras of $L(G)$.

4.2 Blueprints and element orders

Here we give a brief account of [10]; the result from it that we will use is the following theorem.

Theorem 4.8 Let x be a semisimple element of the simply connected form of an exceptional group of Lie type G . If one of the following holds then x is a blueprint for V_{\min} :

- (i) $G = F_4$ and $o(x) > 18$;
- (ii) $G = E_6$, x is real, and $o(x) > 18$;
- (iii) $G = E_7$, $o(x)$ is odd and greater than 75.

Write $v(G)$ for these bounds, so $v(F_4) = v(E_6) = 18$, $v(E_7) = 75$. If V_{\min} extends to a module for \bar{G} , i.e., if \bar{G} doesn't induce a graph automorphism on G , then the same statements hold for \bar{G} , and so any subgroup of \bar{G} containing an element of the appropriate order is a blueprint for V_{\min} . If \bar{G} does induce a graph automorphism on V_{\min} then we need to consider either $L(G)$ or $V_{\min} \oplus V_{\min}^{\tau}$, where τ is a graph automorphism, and whether x is a blueprint on one of these modules. In [10] it is shown that for $G = E_6$ and x real (so every semisimple element in $\mathrm{SL}_2(p^a)$), x is a blueprint for $V_{\min} \oplus V_{\min}^{\tau}$ whenever it is a blueprint for V_{\min} . For F_4 in characteristic 2, [10] proves that there is no almost simple maximal subgroup with socle $\mathrm{SL}_2(2^a)$ for $a \geq 5$ in \bar{G} even if \bar{G} induces a graph automorphism on G , so this case has also been dealt with.

We can push being a blueprint for the minimal module of F_4 up into E_6 and E_7 .

Lemma 4.9 Let G be E_6 or E_7 , and let x be a semisimple element of G that lies in F_4 . If x is a blueprint for the minimal module for F_4 , then x is a blueprint for the minimal modules of E_6 and E_7 , and also the module $V_{\min} \oplus V_{\min}^*$ for E_6 .

Proof: The restrictions of the modules in question to F_4 are a sum of minimal modules and trivial modules. Since 1 is always an eigenvalue of any semisimple element of F_4 on its minimal module, any element of F_4 with the same eigenspaces on the minimal module for F_4 as x also has the same eigenvalues on the three modules mentioned for E_6 and E_7 . This completes the proof. \square

If $G = E_7$ then $v(G) = 75$ is fairly large, and in certain circumstances we can bring this down. Here is one such circumstance.

Proposition 4.10 Let G be the simply connected form of E_7 , and let x be a semisimple element of G . If the 1-eigenspace of x on V_{\min} has dimension at least 6 then x lies inside a conjugate of either an F_4 or A_4 subgroup. If in addition $o(x) > 30$ then x is a blueprint for V_{\min} .

Proof: Since the 1-eigenspace is positive dimensional, x fixes a line on V_{\min} and so lies inside E_6 or B_5 . If x lies inside E_6 then it must fix a line on its minimal module again, and so lies inside F_4 , as claimed, or D_5 . However, D_5 centralizes only a 4-space on V_{\min} , and so x fixes a line on either the natural or 16-dimensional spin module. If x fixes a line on the 10-dimensional module then x lies inside $B_4 \leq F_4$, and if x fixes a line on the 16-dimensional module then x lies inside A_4 , as needed.

Thus we may assume that x lies inside B_5 , whence again it fixes a line on either the 11-dimensional natural module or the 32-dimensional spin module: the first puts x inside D_5 and we are done, and the second puts x inside A_4 again. Thus x lies inside either F_4 or A_4 .

If $x \in F_4$ and $o(x) > 30$ then x is a blueprint for V_{\min} since x is a blueprint for the minimal module for F_4 by Lemma 4.9. On the other hand, if $x \in A_4$ then one uses the proof of the results from [10], noting that the composition factors of A_4 on V_{\min} are, up to multiplicity, 0000, 1000, 0100, 0010 and 0001. A computer program running the algorithm in [10] yields the answer 30. \square

Suppose we want to find the eigenvalues on V_{\min} of semisimple elements of order 63 inside E_7 , which we will need to do when considering $\mathrm{SL}_2(64)$. There are too many to construct them all and store them all effectively, but we can take an element x of order 21 and consider all $3^7 = 2187$ preimages \hat{x} of x in a

torus. Since we have the eigenvalues of all elements of order 21, given a potential multiset of eigenvalues for an element x of order 63 in E_7 , we take the eigenvalues of x^3 , find all semisimple classes of elements of order 21 with those eigenvalues, then consider all preimages of representatives of each of those classes. The eigenvalues of x are valid for coming from E_7 if and only if one of those elements of order 21 has a preimage with those values.

This idea to get the eigenvalues of elements of large composite order will be called the *preimage trick* in the rest of this paper.

4.3 Blueprints inside A_1 s

We now prove that certain semisimple elements, and subgroups of the form $\mathrm{SL}_2(p^a)$ and $\mathrm{PSL}_2(p^a)$ of exceptional groups, are blueprints for a given module by examining the constituents of the restriction of the module to an A_1 subgroup containing the element or subgroup.

The first lemma deals with modules for the algebraic group SL_2 , and when the eigenspaces of semisimple elements match the weight spaces.

Lemma 4.11 Let M be a module for SL_2 with composition factors highest weight modules $L(\lambda_1), \dots, L(\lambda_r)$, arranged so that $\lambda_i \leq \lambda_{i+1}$. Let T be a maximal torus of SL_2 , and let $x \in T$ be a semisimple element of order n . If $\lambda_i < n/2$ then the eigenvalues of x on M are the same as the weight spaces of T . In particular, x is a blueprint for M .

Proof: Since all maximal tori are conjugate, we may assume that x is the matrix

$$\begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix},$$

where ζ is a primitive n th root of 1. The eigenvalues of x on $L(1)$ are ζ, ζ^{-1} , and the eigenvalues of x on $L(\lambda_i)$ are roots of unity $\zeta^{\pm j}$ for $0 \leq j \leq \lambda_i$. If $\lambda_i < n/2$ for all i then the eigenspaces of x are simply the weight spaces of the $L(\lambda_i)$, and so x and T stabilize the same subspaces of M , thus x is a blueprint for M . \square

We will apply this lemma to A_1 subgroups of algebraic groups. We often will end up with composition factors that do not precisely satisfy the hypotheses of this lemma though: if one composition factor has slightly larger highest weight, then although the eigenspaces do not correspond to weight spaces, with some weight spaces being merged, these all take place within one composition factor of the module, and so the finite subgroup $A_1(q)$ of the A_1 is still a blueprint for the module in question, even if the element of order n is not.

Lemma 4.12 Let G be the simply connected form of an exceptional algebraic group of Lie type, and let X be a positive-dimensional subgroup of G of type A_1 . Let x be a semisimple element of X of order n . Let V be a module for G .

- (i) If the composition factors of X on V are $(n/2 - 1)$ -restricted then x and a maximal torus T containing x stabilize the same subspaces of V , so that x is a blueprint for V .
- (ii) Suppose that the highest weights of X on V are $\lambda_1, \dots, \lambda_r$, with $\lambda_i \leq \lambda_{i+1}$, and let $H = A_1(q)$ be a finite subgroup of X containing x . If $\lambda_{r-1} + \lambda_r < n$ then H and X stabilize the same subspaces of V , so that H is a blueprint for V .

Proof: The first part follows immediately from Lemma 4.11, so we concentrate on the second statement. Letting T be a maximal torus of X containing x , if λ and μ are two weights of T on V that are equal when taken modulo n (i.e., yield the same eigenvalue for the action of x), then λ and μ differ by a multiple of n . By assumption on the λ_i , since $\lambda - \mu \geq n$, both λ and μ must be weights for the composition factor $L(\lambda_r)$, since if λ is a weight for one of the other $L(\lambda_i)$ then it lies between $-\lambda_i$ and $+\lambda_i$, and cannot differ by n from any other weight for any other λ_j .

Let W be any H -submodule of V . If W does not contain the factor $L(\lambda_r)$ then the eigenvectors of x on W all come from weight vectors for T , by the previous paragraph, and so T stabilizes W . If W contains $L(\lambda_r)$, then T also stabilizes W by taking duals, as T stabilizes the dual $(V/W)^*$ of V^* . Thus T stabilizes every H -submodule of V , and so $\langle T, H \rangle = X$ and H stabilize the same subspaces of V , as claimed. \square

4.4 Traces for modules of $\mathrm{PGL}_2(p^a)$

Here we produce a specialized result about extending simple modules for $\mathrm{PSL}_2(p^a)$ to $\mathrm{PGL}_2(p^a)$, and the traces and eigenvalues of the elements on such an extension.

There are two extensions of any simple module for $\mathrm{PSL}_2(p^a)$ to $\mathrm{PGL}_2(p^a)$. We will give a way of telling these apart if the dimension of the simple module is odd, which is all that we need in what follows.

If $L(i)$ is a simple module for $\mathrm{PSL}_2(p^a)$ of odd dimension, then of the two extensions of $L(i)$ to $\mathrm{PGL}_2(p^a)$, for one all defining matrices have determinant 1 and for the other half have determinant -1 : to see this, note that all elements in $\mathrm{PSL}_2(p^a)$ act with determinant 1, and the non-trivial 1-dimensional representation acts like -1 for elements outside of $\mathrm{PSL}_2(p^a)$, so a given extension and its product with this 1-dimensional representation give us the two cases. Write $L(i)^+$ for the module for $\mathrm{PGL}_2(p^a)$ for which all matrices have determinant $+1$, and $L(i)^-$ for the other extension. This notation will be used in the proof of the next lemma.

Lemma 4.13 Let p be an odd prime and $a \geq 1$ an integer. Let M be a simple module for $H = \mathrm{PGL}_2(p^a)$ with Brauer character ϕ , and let g be an element of order $p^a \pm 1$. Let t be an involution in $\mathrm{PSL}_2(p^a)$, and let h be the involution in $\langle g \rangle$.

- (i) There are two conjugacy classes of involutions in G . If $o(g)$ is twice an odd number then t and h are representatives of these two classes, and otherwise t and h are conjugate.
- (ii) If $\dim(M)$ is even then $\phi(t) = \phi(h) = 0$.
- (iii) If $\dim(M)$ is odd, then the dimensions of the $+1$ -eigenspace and (-1) -eigenspace differ by 1.
- (iv) If $\dim(M)$ is odd, then $\phi(t) = \pm\phi(h)$. If M has only one of ± 1 as an eigenvalue, then $\phi(t) = \phi(h)$ if and only if either t and h are conjugate, or g has eigenvalue 1 on M .

Proof: (i) That H has two classes of involutions is well known, and one is a class of complements, the other is in $\mathrm{PSL}_2(p^a)$. Thus the second statement follows easily.

- (ii) We use Steinberg's tensor product theorem, lifting all modules to $\mathrm{GL}_2(p^a)$: M has even dimension if and only if, as a tensor product, one of the factors has even dimension, and the Brauer character is 0 for a given element if and only if one of the factors has Brauer character 0 for the same element. Thus we need to check this for the symmetric powers of the natural module $S^i(M')$ for $0 \leq i \leq p-1$, where it is trivial to see that the trace of an involution is 0 on even-dimensional modules and ± 1 on modules of odd dimension.

- (iii) Since $\dim(M)$ is odd and M is self-dual, of course one of ± 1 is an eigenvalue for the action of g . It is an easy exercise to compute the eigenvalues of g on the Steinberg module for $\mathrm{PGL}_2(p^a)$, and we see that these are all distinct if g has order $p^a + 1$, and if g has order $p^a - 1$ then ± 1 appears twice and ∓ 1 appears once.

For other modules, from the definition of the Steinberg module as a tensor product of twists of the fundamental module $L(p-1)$, and the fact that the eigenvalues of g on $L(i)$ all appear in $L(p-1)$, we see that the eigenvalues of g on any simple module appear in the eigenvalues of g on the Steinberg. Thus the result holds since the sum of the dimensions of the $(+1)$ - and (-1) -eigenspaces must be odd.

- (iv) Return to $\mathrm{PGL}_2(p^a)$, and suppose that t and h are not conjugate, so that g has twice odd order. As we saw above, for a fundamental module $L(i)^\pm$ for $0 \leq i \leq p-1$ an even integer, the Brauer character values of $L(i)^+$ on t and h have the same sign, and the Brauer character values of $L(i)^-$ on t and h have opposite signs. Notice that $+1$ is an eigenvalue of M if and only if there are an even number of minus-type modules in the tensor decomposition, and this happens if and only if $\phi(t) = \phi(h)$, as needed.

□

Using this, the following result is now clear.

Corollary 4.14 Let p be an odd prime and $a \geq 1$ an integer. Let G be the simply connected form of E_7 , and let H be a copy of $\mathrm{SL}_2(p^a)$ in G with $Z(G) = Z(H)$. Suppose that g is an element of G such that $o(g) = p^a \pm 1$ is twice an odd number, and -1 is not an eigenvalue for the action of g on $L(G)$. Then the group $\bar{H} = \langle H, g \rangle$ does not satisfy $\bar{H}/Z(H) = \mathrm{PGL}_2(p^a)$.

Proof: Suppose that $\bar{H}/Z(H) = \mathrm{PGL}_2(p^a)$. Let t be an element of H that is an involution in $H/Z(H)$, so $o(t) = 4$. The trace of t on $L(G)$ is -7 or 25 , depending on the class of t in G . The involution h in $\langle g \rangle$ has trace 5 on $L(G)$, since it is an involution in G rather than $G/Z(G)$. We now show that h and t must in fact have the same trace, a contradiction. By Lemma 4.13, any even-dimensional composition factors of $L(G) \downarrow_H$ yield trace 0 for both t and h , and they have the same trace on odd-dimensional composition factors since -1 is not an eigenvalue of g on $L(G)$. Thus the trace of t and h on $L(G)$ is the same, but this is a contradiction.

□

4.5 The graph automorphism of F_4

In this short section we describe how semisimple elements of odd order in F_4 react to the graph automorphism in characteristic 2 . Since the graph automorphism τ does not fix the minimal module V_{\min} , and $L(G)$ has composition factors V_{\min} and V_{\min}^τ , we can see the effect of the graph automorphism on semisimple classes by taking the eigenvalues of an element on x on $L(G)$ and removing those from V_{\min} .

Since the graph automorphism squares to a field automorphism, however, it is slightly more complicated to understand those classes that are left invariant under an outer graph automorphism, since we need to check whether $x^\tau = x^i$ for some i , rather than whether the eigenvalues of x and x^τ match. This is still not difficult using a computer, however; we give two special cases, where a conjugacy class is stable under the graph automorphism (up to powers) and where the classes have integral traces.

Lemma 4.15 Let k be a field of characteristic 2 . Let x be a semisimple element in $G = F_4(k)$ such that x^τ is conjugate to a power of x . If x has order at most 9 , then a power of x has trace on V_{\min} given below.

$o(x)$	Possible traces on V_{\min}
3	-1
5	1
7	$4(\zeta_7 + \zeta_7^{-1}) + 3(\zeta_7^2 + \zeta_7^{-2}) + 5, -(\zeta_7 + \zeta_7^{-1})$
9	$2 - 3(\zeta_9 + \zeta_9^{-1})$

Lemma 4.16 Let k be a field of characteristic 2. Let x be a semisimple element in $G = F_4(k)$ such that the trace of both x and x^τ is an integer, and x and x^τ are not conjugate. If x has order at most 9, then the traces of x and x^τ are as below, where we give x up to graph automorphism.

$o(x)$	Trace of x on V_{\min}	Trace of x^τ on V_{\min}
3	8	-1
5	None	None
7	-2	5
9	-1 (x^3 has trace -1)	2 (x^3 has trace -1)

4.6 \mathfrak{sl}_2 -subalgebras of $L(G)$

In this section we consider subalgebras of $L(G)$, specifically \mathfrak{sl}_2 -subalgebras. The stabilizers of \mathfrak{sl}_2 -subalgebras will be shown to be positive dimensional, and so if a subgroup stabilizes such a subalgebra, it must be contained inside an element of \mathcal{X}^σ .

To begin with, we give a proposition that gives us a criterion for a subgroup H to stabilize an \mathfrak{sl}_2 -subalgebra in the first place. This proposition is a restatement of results of Alexander Ryba from [25], particularly Lemma 10 from that paper.

Proposition 4.17 Let V be a 3-dimensional subspace of $L(G)$, and let H be a subgroup of G such that $HZ(G)/Z(G) = \mathrm{PSL}_2(p^a)$ for some $p \geq 5$. If V is H -stable and a complement for V is also H -stable (i.e., V is a summand of $L(G) \downarrow_H$) and $\mathrm{Hom}_H(V, L(G))$ is 1-dimensional (i.e., there are no other submodules of $L(G) \downarrow_H$ isomorphic to V) then V is a subalgebra of $L(G)$ isomorphic to \mathfrak{sl}_2 .

Proof: Suppose that $L(G) \downarrow_H$ has a unique submodule isomorphic to V , and that this is a summand, so that the quotient $L(G) \downarrow_H / V$ has no quotient isomorphic to $V^* \cong V$. By [25, Lemma 6], we have that V possesses a non-singular trace form, and then we apply Block's theorem [6] to see that V is a simple Lie algebra of type \mathfrak{sl}_2 . \square

In order to use this proposition, we need to know something about \mathfrak{sl}_2 -subalgebras of the Lie algebras of exceptional groups. The following is a theorem of David Stewart and Adam Thomas [28], specialized to the case of $G = E_6, E_7$, for which we need it.

Theorem 4.18 Let $g = E_6$ and $p \geq 7$, or $G = E_7, E_8$ and $p \geq 11$. The \mathfrak{sl}_2 -subalgebras of $L(G)$ are in one-to-one correspondence with the nilpotent orbits of $L(G)$, with a bijection being realized by sending an \mathfrak{sl}_2 -subalgebra to the nilpotent orbit of largest dimension intersecting it non-trivially.

Along with the proof of this theorem, Stewart has representatives for the nilpotent orbits intersecting each of these \mathfrak{sl}_2 s, in a GAP file. When the \mathfrak{sl}_2 is in bijection with a nilpotent class not of order p , there are two nilpotent classes that intersect the \mathfrak{sl}_2 , and for $p \geq 5$ and the \mathfrak{sl}_2 not restricted we give the other class that intersects it for E_6, E_7 and E_8 in Tables 4.3, 4.4 and 4.5 respectively. All of these classes have order

Class	$p = 5$	$p = 7$	$p = 11$
D_4	$3A_1$		
A_5	A_3		
$D_5(a_1)$	$A_2 + A_1$		
$E_6(a_3)$	$A_3 + A_1$		
D_5	A_3	$A_3 + A_1$	
$E_6(a_1)$	$2A_2 + A_1$	A_5	
E_6	$A_2 + 2A_1$	$2A_2 + A_1$	A_5

Table 4.3: Second nilpotent class intersecting an \mathfrak{sl}_2 -subalgebra of $L(E_6)$ for $p \geq 5$ and not restricted

p , and hence e and f lie in different nilpotent orbits of $L(G)$ when the \mathfrak{sl}_2 is not restricted. This yields the following corollary.

Corollary 4.19 Let $g = E_6$ and $p \geq 7$, or $G = E_7, E_8$ and $p \geq 11$. Let H be a copy of $\mathrm{PSL}_2(p)$ in $G/Z(G)$. If $N_{\bar{G}}(H)$ stabilizes an \mathfrak{sl}_2 -subalgebra \mathfrak{h} of $L(G)$, then \mathfrak{h} is restricted and $N_{\bar{G}}(H)$ is contained inside an element of \mathcal{X}^σ .

Proof: If \mathfrak{h} is restricted then it is stabilized by a good A_1 in the algebraic group (see [26]), and $\langle H, A_1 \rangle$ is positive dimensional and stabilizes \mathfrak{h} , so we are done. Thus \mathfrak{h} is not restricted, and therefore e and f lie in different nilpotent classes of $L(G)$.

Since there is a unique conjugacy class of subgroups $\mathrm{PSL}_2(p)$ inside PSL_2 , we see that because the standard $\mathrm{PSL}_2(p)$ inside PSL_2 swaps e and f , H must swap the two nilpotent orbits of \mathfrak{h} , clearly contradicting the fact that they lie inside different orbits of $L(G)$. Hence \mathfrak{h} is restricted, as needed. \square

Class	$p = 5$	$p = 7$	$p = 11$	$p = 13$	$p = 17$
D_4	$(3A_1)'$				
$(A_5)''$	A_3				
$D_4 + A_1$	$4A_1$				
$D_5(a_1)$	$A_2 + A_1$				
$(A_5)'$	A_3				
$A_5 + A_1$	$(A_3 + A_1)'$				
$D_5(a_1) + A_1$	$A_2 + 2A_1$				
$D_6(a_2)$	$D_4(a_1) + A_1$				
$E_6(a_3)$	$(A_3 + A_1)'$				
D_5	A_3	$(A_3 + A_1)'$			
$E_7(a_5)$	$A_3 + A_2$				
A_6	$A_2 + 2A_1$				
$D_5 + A_1$	$(A_3 + A_1)'$	$A_3 + 2A_1$			
$D_6(a_1)$	$A_3 + 2A_1$	$D_4(a_1) + A_1$			
$E_7(a_4)$	$D_4(a_1) + A_1$	$A_3 + A_2$			
D_6	A_3	$A_3 + 2A_1$			
$E_6(a_1)$	$2A_2 + A_1$	$(A_5)'$			
E_6	$A_2 + 2A_1$	$2A_2 + A_1$	$(A_5)'$		
$E_7(a_3)$	$(A_3 + A_1)'$	$D_4(a_1) + A_1$			
$E_7(a_2)$	$2A_2 + A_1$	$A_3 + 2A_1$	$D_6(a_2)$		
$E_7(a_1)$	$A_2 + 2A_1$	$(A_5)'$	D_6	$D_6(a_1)$	
E_7	$A_4 + A_2$	A_6	$A_5 + A_1$	$D_5 + A_1$	D_6

Table 4.4: Second nilpotent class intersecting an \mathfrak{sl}_2 -subalgebra of $L(E_7)$ for $p \geq 5$ and not restricted

Class	$p = 5$	$p = 7$	$p = 11$	$p = 13$	$p = 17$	$p = 19$	$p = 23$	$p = 29$
D_4	$3A_1$							
A_5	A_3							
$D_4 + A_2$	$A_2 + 3A_1$							
$E_6(a_3)$	$A_3 + A_1$							
D_5	A_3	$A_3 + A_1$						
$A_5 + A_1$	$A_3 + A_1$							
$D_5(a_1) + A_2$	$2A_2 + A_1$							
$E_6(a_3) + A_1$	$A_3 + 2A_1$							
$D_5 + A_1$	$A_3 + A_1$	$A_3 + 2A_1$						
$E_8(a_7)$	$A_4 + A_3$							
$A_6 + A_1$	$A_2 + 3A_1$							
$E_6(a_1)$	$2A_2 + A_1$	A_5						
$D_5 + A_2$	$A_3 + A_2$	$A_3 + A_2 + A_1$						
E_6	$A_2 + 2A_1$	$2A_2 + A_1$	A_5					
$D_7(a_2)$	$2A_3$	$A_4 + A_1$						
A_7	$2A_2 + A_1$	A_5						
$E_6(a_1) + A_1$	$2A_2 + 2A_1$	$A_5 + A_1$						
$E_7(a_3)$	$A_3 + A_1$	$D_4(a_1) + A_1$						
$E_8(b_6)$	$D_4(a_1) + A_1$	$E_7(a_5)$						
$D_7(a_1)$	$A_3 + 2A_1$	$A_3 + A_2$						
$E_6 + A_1$	$A_2 + 3A_1$	$2A_2 + 2A_1$	$A_5 + A_1$					
$E_8(a_6)$	$2A_3$	$A_4 + 2A_1$						
D_7	$A_2 + 3A_1$	A_5	$D_5 + A_1$					
$E_8(b_5)$	$2A_2 + 2A_1$	$D_4(a_1) + A_1$	$E_7(a_5)$					
$E_7(a_1)$	$A_2 + 2A_1$	A_5	D_6	$D_6(a_1)$				
$E_8(a_5)$	$2A_2 + A_1$	$E_6(a_3) + A_1$	$E_7(a_4)$					
$E_8(b_4)$	$A_2 + 3A_1$	$A_5 + A_1$	$E_7(a_3)$	$E_7(a_4)$				
$E_8(a_4)$	$A_4 + A_3$	A_5	$D_6(a_2)$	$E_7(a_3)$				
$E_8(a_3)$	$A_4 + A_2 + A_1$	$A_6 + A_1$	$E_6(a_3) + A_1$	$D_6(a_1)$	$E_7(a_3)$			
$E_8(a_2)$	$2A_3$	$A_4 + A_3$	$A_5 + A_1$	D_7	$E_7(a_1)$	$E_7(a_2)$		
$E_8(a_1)$	$2A_2 + 2A_1$	$A_4 + A_2 + A_1$	$D_5(a_1) + A_2$	A_7	D_7	E_7	$E_7(a_1)$	
E_8	$A_3 + A_2 + A_1$	$A_3 + A_2 + A_1$	$A_4 + A_3$	$A_6 + A_1$	A_7	$E_6 + A_1$	D_7	E_7

Table 4.5: Second nilpotent class intersecting an \mathfrak{sl}_2 -subalgebra of $L(E_8)$ for $p \geq 5$ and not restricted. (Missing classes, $D_4 + A_1$, $D_5(a_1)$, $D_5(a_1) + A_1$, $D_6(a_2)$, $E_7(a_5)$, A_6 , $D_6(a_1)$, $E_7(a_4)$, D_6 , $E_7(a_2)$ and E_7 , are exactly as in Table 4.4)

5 Modules for $\mathrm{SL}_2(p^a)$

The purpose of this section is to describe everything we need to know about the simple modules and extensions between them for the groups $\mathrm{SL}_2(p^a)$ for p a prime and $a \geq 1$.

5.1 Modules for $\mathrm{SL}_2(2^a)$

We construct certain modules for $H = \mathrm{SL}_2(2^a)$ for some $a \leq 10$, and prove that various configurations of module do not exist. (The reason we choose $a \leq 10$ is so that these results may be used in work for E_8 , for which $v(E_8) = t(E_8) = 1312$.) The main motivation for this is to get better bounds on the number of certain composition factors that are needed to prevent a particular simple module appearing in the socle of a given module M . The pressure, and more general \mathcal{M} -pressure of Section 2 proves that if M does not fix a line, yet has a trivial composition factor, then it needs at least one more 2-dimensional composition factor than trivial factor. We can do better than this in some circumstances.

We begin with some notation. Let u be an element of order 2 in H . By 2_1 we denote the natural module for H , and define 2_i by the equation

$$2_{i-1}^{\otimes 2} = 1/2_i/1,$$

i.e., 2_i is the twist under the field automorphism of 2_{i-1} . Given this, if I is a subset of $\{1, \dots, a\}$, of cardinality b , we define,

$$2_I^b = \bigotimes_{i \in I} 2_i,$$

for example, $4_{1,2} = 2_1 \otimes 2_2$; the modules 2_I^b for all $I \subseteq \{1, \dots, a\}$ furnish us with a complete set of irreducible modules for H , by Steinberg's tensor product theorem.

We first recall a result of Alperin [1], that determines $\mathrm{Ext}^1(A, B)$ for A, B simple modules for H .

Lemma 5.1 Let A and B be simple H -modules, corresponding to the subsets I and J of $\{1, \dots, a\}$. The dimension of $\mathrm{Ext}^1(A, B)$ is always 0, unless

- (i) $|I \cap J| + 1 = |I \cup J| < a$, and
- (ii) if $i \in I \cup J$ and $i - 1 \notin I \cap J$, then $i - 1 \notin I \cup J$,

and in this case the dimension is 1.

In particular, if $\mathrm{Ext}^1(A, B) \neq 0$ then the dimension of A is either half or double that of B .

Given this, we know that if a module M has a trivial composition factor but does not fix a line, then it has at least two 2-dimensional composition factors, i.e., has positive pressure.

If it has pressure 1, then we can say something about the module still. This is important for F_4 and E_6 because there are no involutions acting projectively there. We will generalize this result in the next lemma, but provide a full proof in this simple case for the benefit of the reader.

Lemma 5.2 Let V be an H -module that has at least one trivial composition factor but no trivial submodules or quotients. If V has pressure 1 then an involution in H acts projectively on V if $\dim(V)$ is even and with a single Jordan block of size 1 if $\dim(V)$ is odd.

Proof: Note that, since V has pressure 1, it cannot have $2_i \oplus 2_j$ or $1^{\oplus 2}$ as a subquotient without fixing a line or hyperplane. We proceed by induction on $\dim(V)$, starting with the even-dimensional case. We may

assume that $\text{soc}(V) = 2_i$ for some i : firstly there are no composition factors of $\text{soc}(V)$ of dimension greater than 2 because the quotient by one would still satisfy the hypotheses of the lemma, and $2_i \oplus 2_j$ cannot be in the socle by the note above.

The module $V/\text{soc}(V)$ has pressure 0, so H must fix a line or hyperplane, but cannot fix a hyperplane by assumption, so $V/\text{soc}(V)$ has a trivial submodule, and it must be unique by the note at the start of the lemma. Quotient out by any possible factors of dimension at least 4 in the socle of $V/\text{soc}(V)$ to get a module W of pressure 0 and with $\text{soc}(W) = 1$. (If there is a 2 in $\text{soc}(W)$ then we find a submodule of pressure 2, not allowed.)

The socle of $W/\text{soc}(W)$ must be 2_j for some j , since $2_j \oplus 2_l$ cannot be a subquotient and 1 only has extensions with simple modules of dimension 2. Now $W/\text{soc}^2(W)$ is again pressure 0, so has a trivial submodule as it cannot have a trivial quotient, and we have constructed a submodule $1/2_j/1$ inside W . Letting U be the quotient of W by this submodule, we have removed $2_i, 2_j, 1^2$ from V , and possibly some other modules, and so an involution acts projectively on U , but it also acts projectively on the kernel of the map $W \rightarrow U$, namely $1/2_j/1$, and on the kernel of the map $V \rightarrow W$ since that has no trivial factors at all, so an involution acts projectively on all of V , as needed.

For odd-dimensional modules, we now simply find any submodule W with a single trivial composition factor and such that it is a quotient of W . The module V/W must have even dimension and has no trivial submodule as otherwise V would have $1 \oplus 1$ as a subquotient. Also, W has pressure 0 since otherwise W with the 1 removed from the top has pressure 2, contradicting Lemma 2.2, and so V/W has pressure 1, thus an involution acts projectively on V/W and with a single 1 on W , as needed. \square

We can generalize this result to modules of larger pressure.

Lemma 5.3 Let V be an H -module that has at least one trivial composition factor but no trivial submodule or quotient. If V has pressure n then an involution in H acts on V with at most n Jordan blocks of size 1.

Proof: As with the previous lemma, choose V to be a minimal counterexample to the lemma, so that the socle and top of V consist solely of 2-dimensional modules. Notice that, by choice of minimal counterexample, there cannot exist a submodule W such that W has no trivial quotients and V/W has no trivial submodules, since otherwise one of W and V/W would also be a counterexample.

Let W be a minimal submodule with a trivial composition factor but no trivial quotient, and note that, since all simple modules with non-trivial 1-cohomology have dimension 1, W contains a single trivial composition factor. If $W = V$ then V itself must have a single trivial composition factor, and so V has a single block of size 1, and the result holds since V must have pressure at least 1.

Thus $W < V$. If V/W has no trivial submodule then we have a contradiction by the statements above, so $\text{soc}(V/W)$ has a trivial composition factor: let W_2 denote the preimage of this trivial submodule of $\text{soc}(V/W)$ in V . We claim that W_2 has a quotient $1/2/1$. If this true then, since W_2 has pressure at least 0 and u acts projectively, and V/W_2 has no trivial submodules or quotients, there must be at most n Jordan blocks of size 1 in V , a contradiction. \square

Lemma 5.4 Let $a = 3$. If M is an even-dimensional module with $2n > 0$ trivial composition factors and no trivial submodule or quotient, then it has at least $3n$ composition factors of dimension 2.

Proof: Note that if $M = M_1 \oplus M_2$ with the M_i even-dimensional, then by induction M satisfies the conclusion of the lemma: thus M is either indecomposable or the sum of two odd-dimensional indecomposable modules.

The projective cover of 2_1 is

$$2_1/1, 4_{1,3}/2_1, 2_2, 2_3/1, 1, 4_{2,3}/2_1, 2_2, 2_3/1, 4_{1,3}/2_1.$$

Remove any 4-dimensional factors from the top and socle of M , so that M is a submodule of a sum of copies of projectives $P(2_i)$. If M has seven socle layers then $P(2_i)$ is a summand of M , so $M = P(2_i)$ and we are done. Thus M has at most five socle layers. The number of 2s in the first and third socle layers must be at least as many as the number of 1s in the second layer, and there are at least as many 2s in the third and fifth socle layers as 1s in the fourth layer. We therefore must have that there are at least $3n$ 2-dimensional factors in M , as claimed. \square

Our next result is the best result possible in this direction.

Lemma 5.5 Let $H = \mathrm{SL}_2(2^a)$ for $3 \leq a \leq 10$. The largest submodule of $P(2_1)$ whose composition factors have dimension 1 or 2 has dimension 10, and structure

$$2_1/1/2_2, 2_a/1/2_1,$$

where 2_a is a quotient of this module.

Let M be a module for $\mathrm{SL}_2(2^a)$ for some $a \leq 10$. Suppose that u acts on M with b Jordan blocks of size 1. If there are $c > 0$ trivial composition factors in M , but no trivial submodules or quotients, then M has at least $2b + 3(c - b)/2$ composition factors of dimension 2.

Proof: The first statement, of the structure of the largest submodule of $P(2_1)$ with factors of dimension 1 and 2, is easily verified by computer in the range specified. Thus we concentrate on the second statement, which we prove by induction on $\dim(M)$. By removing all submodules and quotients of simple modules of dimension at least 4, we may assume that M is a submodule of a sum of $P(2_i)$ s for various i .

The largest submodule N of M with factors of dimension 1 and 2, firstly has no trivial submodules or quotients, and secondly is a submodule of sums of modules of the form in the first part of this lemma. Since u acts on the subquotient $1/2/1$ with two blocks of size 2, if u has b' Jordan blocks of size 1 on N and c' trivial composition factors in total, then we see that modulo the third socle $\mathrm{soc}^3(N)$ of N we must have exactly $(c' - b')/2$ trivial composition factors; hence we have $(c' - b')/2$ 2-dimensionals in the fifth socle layer, and the socle and third socle layers have at least $(c' + b')/2$ 2-dimensional modules each, yielding $2b' + 3(c' - b')/2$ 2-dimensionals in total. The quotient M/N also has no trivial submodules or quotients, so we are done if the number of Jordan blocks of size 1 in the action of u on M/N is at least $b - b'$, but this is clear. This completes the proof. \square

Obviously this improves the result that the pressure must be positive, i.e., if there are c trivials then there must be at least $(c + 1)$ 2-dimensionals; we easily see from the modules in the lemma above that this bound of $(3c + b)/2$ is best possible.

Lemma 2.3 shows that, not only can we not fix a 1-space on either V_{\min} or $L(G)$, but we cannot fix a 2-space on V_{\min} either for F_4 , E_6 and E_7 . By Lemma 5.1 we see that 2s only have non-split extensions with 1 and 4s, so we would like a similar result to the previous one, counting the number of 4-dimensionals in a module M that has 2-dimensional composition factors but no 2-dimensional submodules or quotients. We start with the easier case, where there are no trivial composition factors in M at all. Notice that we can use \mathcal{M} -pressure here as well, but we can do a bit better using the structure of modules for $\mathrm{SL}_2(2^a)$.

(We do not need to consider $a > 6$ here as these lemmas are not of use for E_8 .)

Lemma 5.6 Let $H = \mathrm{SL}_2(2^a)$ for $4 \leq a \leq 6$. The largest submodule of $P(4_{i,j})$ whose composition factors have dimension 2 and 4 is as follows: for $j = i \pm 1$, we have a 10-dimensional module

$$4_{i-1,i+1}/2_{i+1}/4_{i,i+1};$$

for $a = 4$ we have a 28-dimensional module

$$4_{1,3}, 4_{1,3}/2_1, 2_3/4_{1,4}, 4_{2,3}/2_1, 2_3/4_{1,3};$$

for $j = i \pm 2$ and $a > 4$ we have a 32-dimensional module

$$4_{i,i+2}, 4_{i,i+2}/2_i, 2_{i+2}/4_{i+1,i+2}, 4_{i,i+3}, 4_{i-1,i+2}/2_i, 2_{i+2}/4_{i,i+2},$$

with $4_{i-1,i+2}$ as a quotient. In all other cases, we have the module

$$4_{i,j}, 4_{i,j}/2_i, 2_j/4_{i,j-1}, 4_{i,j+1}, 4_{i+1,j}, 4_{i-1,j}/2_i, 2_j/4_{i,j},$$

with $4_{i,j-1}$ and $4_{i-1,j}$ as quotients.

Consequently, if M is a module with no trivial composition factors, with $c > 0$ composition factors of dimension 2, and no 2-dimensional submodule or quotient, then M has at least c composition factors of dimension 4.

Proof: The proof follows that of Lemma 5.5: we cannot produce a module $4/2, 2/4/2, 2/4$ since the 4s in the middle of the modules above do not have extensions with both 2s by Lemma 5.1. Thus we have at least $4/2, 2/4, 4/2, 2/4$, and so we need as many 4s as 2s. \square

Of course, unlike in the case of Lemma 5.5, the $4_{i,j}$ s are not all the same up to field automorphism, and so the largest constructible modules depend on which 4s and 2s we have.

The next lemma brings together the previous two results, in the sense that we want to know how many 1s and 2s we can stack on top of a given simple module of dimension 4. This lemma gives that answer, and hence how many 4s one needs to hide all 1 and 2s inside the middle of the module.

Lemma 5.7 Let $H = \mathrm{SL}_2(2^a)$ for some $2 \leq a \leq 6$. The largest submodule of $P(4_{1,2})$ whose composition factors modulo the socle have dimensions 1 or 2 is

$$2_2/1/2_3/1/2_2/4_{1,2},$$

and an involution acts projectively on this module.

For $a = 4$ and $a \geq 5$ we have

$$2_2, 2_4/1/2_1, 2_3/4_{1,3}, \quad \text{and} \quad 2_2/1/2_1, 2_3/4_{1,3}$$

respectively. For $a \geq 6$ and $i = 4, 5$ we have $1/2_1, 2_i/4_{1,i}$.

In particular, if M is a module for H with no trivial or 2-dimensional submodules or quotients, and it has $2n$ trivial composition factors for some $n > 0$, then it has $n' \geq n + 1$ 4-dimensional factors, and between $2n + 1$ and $4n'$ 2-dimensional composition factors.

Proof: The facts about the largest submodule of $P(4_{i,j})$ can easily be checked with a computer. For the conclusion, we proceed by induction. By removing submodules and quotients of dimension 8 and above, we may assume that the socle and top of M consists entirely of 4-dimensional modules.

Let $M_1 = \text{soc}(M)$ and M_2/M_1 be the $4'$ -radical of M/M_1 , noting that M_2/M_1 is the direct sum of its $\{1, 2\}$ -radical and $\{1, 2, 4\}'$ -radical. By the first part, if M_2 has $2m$ trivial modules then M_1 has at least m copies of 4-dimensional modules to suppose the $2m$ trivials, and from the structure above the number of 2-dimensionals is at most $4m$. Applying induction to M/M_2 , this proves there are at least $n + 1$ different 4-dimensional factors and at most $4n$ 2-dimensional factors; there are at least $2n + 1$ 2-dimensional factors since M must have positive pressure. \square

5.2 Modules for $\text{SL}_2(3^a)$

In this section we describe the simple modules for $H = \text{SL}_2(3^a)$ for $a \leq 7$, describe various extensions between some of the simple modules, and prove the existence or non-existence of various indecomposable modules.

Let $L = \text{SL}_2(3) \leq H$. The simple modules for L have dimension 1, 2 and 3, with only the 2 being faithful. Therefore, the modules for H are tensor products of modules of dimension 2 and 3, with a module of dimension $2^m 3^n$ being faithful if and only if m is odd.

Writing 2_i for the image of 2 under i iterations of the Frobenius map, and similarly for 3_i , the simple modules for H can be labelled by $2^m 3^n_{r_1, \dots, r_{m+n}}$, where $m, n \geq 0$ are integers, $\{r_1, \dots, r_{m+n}\} \subset \{1, \dots, a\}$ with the r_i distinct, with

$$2^m 3^n_{r_1, \dots, r_{m+n}} = \left(\bigotimes_{j=1}^m 2_{r_j} \right) \otimes \left(\bigotimes_{j=m+1}^{m+n} 3_{r_j} \right).$$

Hence for example $12_{2,3,1} = 2_2 \otimes 2_3 \otimes 3_1$ is a module for $\text{PSL}_2(3^a)$ for any $a \geq 3$.

We need to understand the restrictions of these simple modules to L , in order to understand which ones we can have in the restrictions of minimal modules for $G = F_4, E_6, E_7$.

Lemma 5.8 Let $H = \text{PSL}_2(3^a)$, $a \geq 1$, and let M be a simple module of dimension at most 56. The restriction of M to $\text{PSL}_2(3)$ is as below.

Module	Restriction	Composition factors of restriction
1	1	1
3	3	3
$4 = 2 \otimes 2$	$3 \oplus 1$	$3, 1$
$9 = 3 \otimes 3$	$3^{\oplus 2} \oplus P(1)$	$3^2, 1^3$
$12 = 2 \otimes 2 \otimes 3$	$3^{\oplus 3} \oplus P(1)$	$3^3, 1^3$
$16 = 2 \otimes 2 \otimes 2 \otimes 2$	$3^{\oplus 4} \oplus P(1) \oplus 1$	$3^4, 1^4$
$27 = 3 \otimes 3 \otimes 3$	$3^{\oplus 7} \oplus P(1)^2$	$3^7, 1^6$
$48 = 2 \otimes 2 \otimes 2 \otimes 2 \otimes 3$	$3^{\oplus 12} \oplus P(1)^{\oplus 4}$	$3^{12}, 1^{12}$

We now move on to extensions. With the labelling above, we have the following easy lemma, which can be found for example in [3].

Lemma 5.9 For any $a > 1$, a simple module M has non-trivial 1-cohomology if and only if $M = 4_{i,i+1}$ for some $1 \leq i \leq a$, and

$$\dim(\text{Ext}^1(1, 4_{i,i+1})) = \begin{cases} 1 & a \geq 3, \\ 2 & a = 2. \end{cases}$$

We will need more detailed information about extensions between low-dimensional modules for H , and we summarize that which we need now. We restrict to the case when $a \neq 2$, because in this case things are slightly different, with that pesky 2-dimensional 1-cohomology group, and secondly because we describe the full projectives for this group after the lemma anyway.

Lemma 5.10 Let $H = \mathrm{PSL}_2(3^a)$ for $3 \leq a \leq 6$. The following extension groups have dimension 1, for all $1 \leq i, j \leq a$:

$$(4_{i,i+1}, 1), \quad (1, 4_{i,i+1}), \quad (3_i, 4_{i-1,i}), \quad (4_{i-1,i}, 3_i), \quad (4_{i,j}, 4_{i\pm 1,j}), \quad (4_{i,j}, 4_{i,j\pm 1}),$$

If A and B are simple modules for H of dimension at most 9 then $\mathrm{Ext}^1(A, B) = 0$ unless (A, B) is on the list above.

We now consider certain modules. For $a = 2$, the structures of the projective indecomposable modules are as follows:

$$\begin{array}{ccc} 1 & 3_i & 4 \\ 4 & 4 & 1 & 1 & 3_1 & 3_2 \\ 1 & 1 & 1 & 3_1 & 3_2 & 1 & 3_{3-i} & 4 & 4 & 4 \\ 4 & 4 & 4 & 1 & 1 & 3_1 & 3_2 \\ 1 & 3_i & 4 \end{array}$$

We see that if a module M has five socle layers then it has a projective summand. More generally, if M has trivial composition factors, then we can use these to prove that M must have more 4s than pressure arguments suggest.

Lemma 5.11 Let $H = \mathrm{PSL}_2(9)$ and let M be a module for H . If M has no trivial submodules or quotients, and there are $2n - 1$ or $2n$ trivial composition factors, then the number of 4-dimensional factors is at least $2n$.

Furthermore, the only submodules of $P(4)$ consisting of 4s and 1s are submodules of a self-dual module $4/1, 1/4$. In particular, there is no uniserial module of the form $4/1/4$.

Proof: Let M be a module for H . We may assume that M is indecomposable. If M is the 9-dimensional projective simple then the claim is true. If M has any 3-dimensional submodules or quotients then we may remove them without affecting the claim, and so we may assume that M is a submodule of copies of $P(4)$.

If M is projective then the result holds, so M is not projective, in which case it has at most four socle layers. Since the fourth socle layer consists solely of 1s and 3_i s, M must actually have three socle layers. In particular, the trivials are all in the second socle layer, so if there are $2n - 1$ or $2n$ of them, there must be at least n copies of the 4-dimensional module in the socle, and similarly in the top. This completes the proof of the first claim.

The second is easy to see by a computer proof that $4/1, 1/4$ is the largest such module. Since it is self-dual, we cannot construct a $4/1/4$ inside it, yielding the second statement. \square

Lemma 5.12 Let $H = \mathrm{PSL}_2(3^a)$ for some $2 \leq a \leq 7$. There does not exist a uniserial module with socle layers $4_a, 1$ and 4_b , where 4_a and 4_b are any of the simple modules of dimension 4.

As a consequence, if $a \neq 2$ and M is a module with composition factors of dimension $4^{2i-\alpha}, 1^i$ for some $i > 0$ and $\alpha \geq 0$, then M has a trivial submodule or quotient.

In particular, we see that once we have constructed $p(p-1)$ non-projective, indecomposable modules for H then we must have found them all. Thus we start with the simple and projective modules for H : letting $M = L(1)$ be the natural module for H , we construct all simple modules using symmetric powers

$$L(i) = S^i(M) \quad 0 \leq i \leq p-1,$$

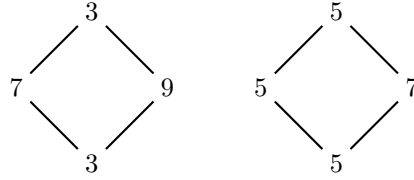
with $L(i)$ being of dimension $i+1$. As with the case of $\mathrm{SL}_2(2^a)$, we will normally write the single number i to refer to the simple module of dimension i , and so a module $3/5$ for $\mathrm{SL}_2(7)$, for example, is an 8-dimensional module with 5-dimensional socle $L(4)$ and 3-dimensional top $L(2)$. The odd-dimensional simple modules are modules for $\mathrm{PSL}_2(p)$, and the even-dimensional ones are faithful modules for $\mathrm{SL}_2(p)$.

Having defined the simple modules, we consider the projectives: the Steinberg module $L(p-1)$ of dimension p is already projective, and for each simple module i with $1 \leq i \leq p-1$, the projective module $P(i)$ has structure

$$i/(p+1-i), (p-1-i)/i,$$

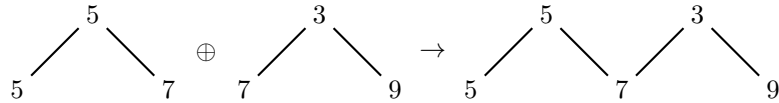
except when $i = 1$, in which case $p+1-i$ would have dimension p , and we have $1/(p-1)/1$, and when $i = p-1$, so $p-1-i$ would have dimension 0, and we have $(p-1)/2/(p-1)$.

We represent these in diagrams, with lines linking two composition factors A and B if there is a non-split extension A/B as a subquotient of the module. For example, here are $P(3)$ and $P(5)$ for $\mathrm{PSL}_2(11)$.



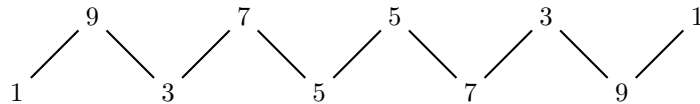
Using these we construct indecomposable modules as follows: we have modules of the form $i/(p+1-i)$ and $i/(p-1-i)$, and also two modules of the form $i/(p-1-i), (p+1-i)$ and $(i+2)/(p-1-i)$. These two indecomposables can be summed together, then quotiented by a diagonal submodule $p-1-i$ to make a new module with four composition factors.

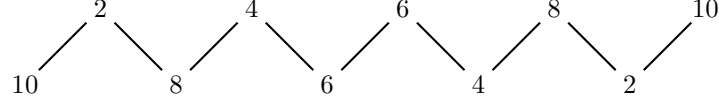
It is easier to visualize using diagrams. In the example above, we can remove the socles of the two projectives to get modules $3/7, 9$ and $5/5, 7$, take their direct product, and then quotient out by a diagonal 7.



This process certainly produces a module, with quotients both of our original summands, and so this module must be indecomposable. Note that if one tries to do this with say two copies of $3/7, 9$ then the fact that $\mathrm{Ext}^1(3, 7)$ is 1-dimensional means that this module splits, so one needs the modules at the top (in this case 3 and 5) to be different.

One can continue this process until one constructs an indecomposable module M with all (non-projective) simple modules appearing in the top and the socle of M exactly once. As an example, the diagrams of the two such modules for $p = 11$ (one for $\mathrm{PSL}_2(11)$, one for faithful modules for $\mathrm{SL}_2(11)$) are as follows:





We can take subquotients of these modules and construct new indecomposable modules, and we claim that this constructs all non-projective, indecomposable modules for $\mathrm{SL}_2(p)$.

Firstly, the non-simple indecomposable subquotients of the module M are in one-to-one correspondence with connected subdiagrams of the diagram with at least one edge, since one notes that no two distinct subdiagrams of the diagram above have the same first and second rows. In the case of simple modules, of course each appears twice.

The number of connected subdiagrams of each diagram with at least one edge is $(p-1)(p-2)/2$ (i.e., we choose the start and end points), and add in the $p-1$ simple modules (other than the Steinberg), and the $p-1$ non-simple projective modules, to get

$$(p-1)(p-2) + 2(p-1) = p(p-1).$$

This is the number of indecomposable modules for the normalizer, and so we must have constructed all indecomposable modules for $\mathrm{SL}_2(p)$.

It is clear from this ‘zigzag’ structure, that for any indecomposable module, the socle is a collection of modules, and if A and B lie in the socle so does any module with dimension between $\dim(A)$ and $\dim(B)$. We have proved the following proposition.

Proposition 5.14 Let $H = \mathrm{SL}_2(p)$, and let M be an indecomposable module for H .

- (i) If M has one socle layer then M is simple, and there are p such modules, one of each dimension.
- (ii) If M has three socle layers then $M = P(i)$ for some $1 \leq i \leq p-1$.
- (iii) If M has two socle layers then the socle of M consists of modules of dimension $i, i+2, \dots, j$ ($i \leq j$), and the top consists of modules $p-j+\epsilon, p-j+\epsilon+2, \dots, p-i+\delta$, where $\epsilon, \delta = \pm 1$. There are $(p-1)(p-2)$ such modules.

The indecomposable modules for $\mathrm{PSL}_2(7)$ other than the Steinberg are below, ordered so that the modules in column i have dimension congruent to i modulo 7.

1	3, 5/3, 5	3	1, 3, 5/1, 3, 5	5	3/3	$P(1)$
3/5	1, 3/5	1, 3, 5/3, 5	3/3, 5	1, 3/3, 5	1/5	$P(3)$
5/3	5/1, 3	3, 5/1, 3, 5	3, 5/3	3, 5/1, 3	5/1	$P(5)$

As another example, the indecomposable modules for $\mathrm{PSL}_2(5)$ are

$$1, 3, 1/3, 3/1, P(1), P(3), 3, 1/3, 3/1, 3, 3/3, 1, 3/1, 3.$$

Since each of these modules M is in Green correspondence with an indecomposable module V of dimension at most $p-1$, and Green correspondence means that the restriction of M to $N_H(P)$ is a sum of V and projective modules, this means that a unipotent element u of order p in H acts on M with at most one Jordan block of size not equal to p (if M is projective then all blocks have size p , of course). Notice that the only indecomposable modules with no Jordan blocks of size p in the action of u are either simple or of dimension $p-1$.

The next lemma is an easy consequence of these facts.

Lemma 5.15 Let $H = \mathrm{PSL}_2(p)$, and let M be a module for H over a field of characteristic p . Let $u \in H$ have order p . If n_i denotes the number of Jordan blocks of size i in the action of u on M , then

$$n_p \geq \sum_{1 \leq i < (p-1)/2} n_{2i}.$$

Proof: The modules of dimension at most p are either simple, so u acts on them with a block of odd size, or are $i/(p-1-i)$, so u acts with a single block of size $p-1$. Therefore if there is a block of even size $i < p-1$ then it must come from an indecomposable module of dimension greater than p , and so we get at least one block of size p , as needed. \square

For $G = E_7$ we also must consider $H = \mathrm{SL}_2(p)$ with $Z(H) = Z(G)$. In this case we want a similar result to the above but for faithful modules.

Lemma 5.16 Let $H = \mathrm{SL}_2(p)$, and let M be a module for H over a field of characteristic p on which the central involution z acts as the scalar -1 . Let $u \in H$ have order p . If n_i denotes the number of Jordan blocks of size i in the action of u on M , then

$$n_p \geq \sum_{1 \leq i < (p-1)/2} n_{2i-1}.$$

Proof: Similar to Lemma 5.15, and omitted. \square

We often want to understand self-dual modules for H , since the minimal modules V_{\min} are self-dual for F_4 and E_7 , and the adjoint module $L(G)$ is always self-dual. Using the statements above, if n_i is odd, where again n_i is the number of blocks of size i in the action of u , there must be a self-dual indecomposable summands of dimension congruent to i modulo p .

The next lemma follows from Proposition 5.14 and classifies self-dual indecomposable modules for $\mathrm{SL}_2(p)$. From our zigzag diagrams above, it is clear which the self-dual modules are: choose the same simple module as the start and end points of the subdiagram.

Lemma 5.17 Let $H = \mathrm{SL}_2(p)$, and let M be a self-dual indecomposable module for H . If M is not simple or projective, then M has socle (and top) consisting of pairwise non-isomorphic modules N_1, N_2, \dots, N_r , where $\dim(N_i) - \dim(N_{i-1}) = 2$ and $\dim(N_1) + \dim(N_r) = p \pm 1$. In particular, there are exactly $p-1$ non-projective, indecomposable self-dual modules for $\mathrm{PSL}_2(p)$, and exactly $p-1$ non-projective, indecomposable and faithful self-dual modules for $\mathrm{SL}_2(p)$.

Therefore, if $p \equiv 1 \pmod{4}$, all non-projective indecomposable self-dual modules for $\mathrm{PSL}_2(p)$ have dimension congruent to an odd number modulo p , and all non-projective, indecomposable, faithful self-dual modules for $\mathrm{SL}_2(p)$ have dimension congruent to an even number modulo p .

We can use this to get a better handle on which possible Jordan block structures a given unipotent element u can have, given it lies inside a copy of $\mathrm{PSL}_2(p)$ for $p \equiv 1 \pmod{4}$. We split the result into two corollaries depending on whether one has modules for $\mathrm{PSL}_2(p)$ or $\mathrm{SL}_2(p)$.

Corollary 5.18 Let $H = \mathrm{PSL}_2(p)$ with $p \equiv 1 \pmod{4}$, and let M be a self-dual module for L . Let u be an element of order p in H . The action of u has an even number of blocks of a given even size i , and there are at least as many blocks of size p as there are blocks of size all even numbers less than $p-1$.

Corollary 5.19 Let $H = \mathrm{SL}_2(p)$ with $p \equiv 1 \pmod{4}$, and let M be a self-dual module for H on which the central involution z acts as the scalar -1 . Let u be an element of order p in H . The action of u has an even number of blocks of a given odd size i , and there are at least as many blocks of size p as there are blocks of size all odd numbers less than p .

We now turn to tensor products. By Steinberg's tensor product theorem, simple modules for $\mathrm{SL}_2(p^a)$ are tensor products of Frobenius twists of p -restricted modules, i.e., $L(i)$ for $i \leq p-1$. These restrict to $\mathrm{SL}_2(p)$ as tensor products of simple modules, so it will come in handy to understand the tensor products of simple modules for $\mathrm{SL}_2(p)$.

The next result gives the tensor product of any two simple modules for L , and will be of great use when computing the restriction of simple $\mathrm{SL}_2(p^a)$ -modules to $\mathrm{SL}_2(p)$.

Proposition 5.20 Let $H = \mathrm{SL}_2(p)$. If $0 \leq \mu \leq \lambda \leq p-1$ then $L(\lambda) \otimes L(\mu)$ is given by one of the following:

(i) If $\lambda + \mu < p$ then

$$L(\lambda) \otimes L(\mu) = L(\lambda - \mu) \oplus L(\lambda - \mu + 2) \oplus \cdots \oplus L(\lambda + \mu - 2) \oplus L(\lambda + \mu).$$

(ii) If $\lambda + \mu \geq p$ and $\lambda < p-1$ then

$$\begin{aligned} L(\lambda) \otimes L(\mu) &= L(\lambda - \mu) \oplus L(\lambda - \mu + 2) \oplus \cdots \oplus L(a) \\ &\oplus \begin{cases} P(\lambda + \mu) \oplus P(\lambda + \mu - 2) \oplus \cdots \oplus P(p+1) \oplus L(p-1) & \mu \text{ even} \\ P(\lambda + \mu) \oplus P(\lambda + \mu - 2) \oplus \cdots \oplus P(p) & \mu \text{ odd} \end{cases} \end{aligned}$$

where $a = 2p - (\lambda + \mu + 4)$.

(iii) $L(p-1) \otimes L(p-1) = P(1) \oplus P(3) \oplus \cdots \oplus P(p-1)$.

This result can be found, for example, in [12] and explicitly in [11, Lemma 3.1].

5.4 Modules for $\mathrm{SL}_2(p^a)$ for $p \geq 5$ and $a > 1$

As with modules for $\mathrm{SL}_2(3^a)$ we need a notation system for the simple modules, and as in that section, we let 2_1 denote the natural module, $i_1 = S^{i-1}(2_1)$ for $2 \leq i \leq p$ be the symmetric powers (the p -restricted modules) and let i_{j+1} denote the application of the Frobenius morphism to i_j . We then write, for module of dimension n formed as the tensor product of m twisted fundamental modules, n_{a_1, \dots, a_m} , in order of increasing dimension of factor; for example, the module $2_1 \otimes 3_2$ will be denoted $6_{1,2}$, and $3_1 \otimes 3_2 \otimes 2_3$ will be denoted $18_{3,1,2}$. If the integer n has a unique decomposition as a product of exactly m integers greater than 1 such that the module would be for the correct group (i.e., $\mathrm{PSL}_2(p^a)$ or $\mathrm{SL}_2(p^a)$) then we simply write that, so that 6_1 and $6_{1,2}$ for $\mathrm{PSL}_2(49)$ are unambiguous. Sometimes there are modules that could be either for SL_2 or PSL_2 , such as $12_{1,2}$ for $p \geq 7$, which is either $2_1 \otimes 6_2$ or $3_1 \otimes 4_2$, but context will tell us which. When there genuinely is ambiguity, for example, $18_{1,2}$ when $p \geq 11$, as it could be $2_1 \otimes 9_2$ or $3_1 \otimes 6_2$, we label them with subscripts $18_{1,2}^{(1)}$ and $18_{1,2}^{(2)}$ according to the lexicographic ordering on the partitions of 18, but in these rare cases we remind the reader which is which.

We start with some restrictions of simple $\mathrm{PSL}_2(q)$ -modules to $\mathrm{PSL}_2(p)$. This is needed because we often understand the action of $\mathrm{PSL}_2(p)$ on the minimal or adjoint modules completely. We consider modules of dimension at most 56 to include the minimal modules of F_4 , E_6 and E_7 . We use Proposition 5.20 to compute

the restrictions of modules for $\mathrm{SL}_2(p^a)$ to $\mathrm{SL}_2(p)$. Since $v(E_7) = 75$ we only list restrictions for $p^a \leq 150$ and dimension up to 56.

Lemma 5.21 Let $H = \mathrm{PSL}_2(5^a)$ for $a = 2, 3$, and let M be a simple module of dimension at most 56. The restriction of M to $L = \mathrm{PSL}_2(5)$ is as below.

Module	Restriction	Composition factors of restriction
1	1	1
3	3	3
$4 = 2 \otimes 2$	$3 \oplus 1$	3, 1
5	5	5
$8 = 2 \otimes 4$	$5 \oplus 3$	5, 3
$9 = 3 \otimes 3$	$1 \oplus 3 \oplus 5$	5, 3, 1
$12 = 2 \otimes 2 \otimes 3$	$5 \oplus 3^{\oplus 2} \oplus 1$	5, 3^2 , 1
$15 = 3 \otimes 5$	$5 \oplus P(3)$	5, 3^3 , 1
$16 = 4 \otimes 4$	$5 \oplus P(3) \oplus 1$	5, 3^3 , 1^2
$20 = 2 \otimes 2 \otimes 5$	$5^{\oplus 2} \oplus P(3)$	5^2 , 3^3 , 1
$24 = 2 \otimes 4 \otimes 3$	$5^{\oplus 2} \oplus P(3) \oplus 3 \oplus 1$	5^2 , 3^4 , 1^2
$25 = 5 \otimes 5$	$5^{\oplus 2} \oplus P(3) \oplus P(1)$	5^2 , 3^4 , 1^3
$27 = 3 \otimes 3 \otimes 3$	$5^{\oplus 2} \oplus P(3) \oplus 3^{\oplus 2} \oplus 1$	5^2 , 3^5 , 1^2
$40 = 2 \otimes 4 \otimes 5$	$5^{\oplus 3} \oplus P(3)^{\oplus 2} \oplus P(1)$	5^3 , 3^7 , 1^4
$45 = 3 \otimes 3 \otimes 5$	$5^{\oplus 4} \oplus P(3)^{\oplus 2} \oplus P(1)$	5^4 , 3^7 , 1^4
$48 = 4 \otimes 4 \otimes 3$	$5^{\oplus 3} \oplus P(3)^{\oplus 2} \oplus P(1) \oplus 3$	5^4 , 3^8 , 1^4

Consequently, if V is a module for H of dimension at most 56 such that $V \downarrow_L$ has more trivial than 3-dimensional composition factors, then H stabilizes a line on V .

Proof: We prove the last statement: from the table above we see that only the trivial has more 1s than 3s in its restriction to L . Suppose that the composition factors of $V \downarrow_L$ are $5^i, 3^j, 1^k$, with $k > j$. If there are α trivial factors and β 8-dimensional factors in V , then $\alpha \geq k - (j - \beta) > \beta$, so V has negative pressure and so H fixes a line on V , as needed. \square

For $\mathrm{PSL}_2(25)$, we will need the eigenvalues of an element of order 12 on the simple modules, so we list them here. These are of course easy to compute.

Lemma 5.22 Let $H = \mathrm{PSL}_2(25)$, and let x be a semisimple element of order 12 in H . Let ξ denote a primitive 12th root of unity. Choosing ξ so that x acts on the symmetric square of the natural module for $\mathrm{SL}_2(25)$ with eigenvalues $\xi^{\pm 1}$, the eigenvalues of x on the various simple modules for H are as follows.

Dimension	Eigenvalues
1	1
3	$1, (\xi, \xi^{11})/(\xi^5, \xi^7)$
4	$(\xi^2, \xi^{10}), (\xi^3, \xi^9)$
5	$1, (\xi^2, \xi^{10}), (\xi, \xi^{11})/(\xi^5, \xi^7)$
8	$(\xi^2, \xi^{10}), (\xi^3, \xi^9), (\xi^4, \xi^8), (\xi, \xi^{11})/(\xi^5, \xi^7)$
9	$1, (-1)^2, (\xi, \xi^{11}), (\xi^4, \xi^8), (\xi^5, \xi^7)$
15	$1, (-1)^2, (\xi, \xi^{11}), (\xi^2, \xi^{10}), (\xi^3, \xi^9), (\xi^4, \xi^8), (\xi^5, \xi^7), (\xi, \xi^{11})/(\xi^5, \xi^7)$
16	$(-1)^2, (\xi, \xi^{11}), (\xi^2, \xi^{10}), (\xi^3, \xi^9)^2, (\xi^4, \xi^8)^2, (\xi^5, \xi^7)$
25	$1^3, (-1)^2, (\xi, \xi^{11})^2, (\xi^2, \xi^{10})^2, (\xi^3, \xi^9)^2, (\xi^4, \xi^8)^2, (\xi^5, \xi^7)^2$

Here, $(\xi, \xi^{11})/(\xi^5, \xi^7)$ means either (ξ, ξ^{11}) or (ξ^5, ξ^7) , depending on the isomorphism type of the module.

We now give the analogue of Lemma 5.21 for $p = 7$. Again, we only need go up to $p^a = 150$, so just 49 in this case.

Lemma 5.23 Let $H = \text{PSL}_2(49)$, and let M be a simple module for H . The restriction of M to $L = \text{PSL}_2(7)$ is as below.

Module	Restriction	Composition factors of restriction
1	1	1
3	3	3
$4 = 2 \otimes 2$	$3 \oplus 1$	3, 1
5	5	5
7	7	7
$8 = 2 \otimes 4$	$5 \oplus 3$	5, 3
$9 = 3 \otimes 3$	$5 \oplus 3 \oplus 1$	5, 3, 1
$12 = 2 \otimes 6$	$7 \oplus 5$	7, 5
$15 = 3 \otimes 5$	$7 \oplus 5 \oplus 3$	7, 5, 3
$16 = 4 \otimes 4$	$7 \oplus 5 \oplus 3 \oplus 1$	7, 5, 3, 1
$21 = 3 \otimes 7$	$7 \oplus P(5)$	7, 5^2 , 3, 1
$24 = 4 \otimes 6$	$7 \oplus P(5) \oplus 3$	7, 5^2 , 3^2 , 1
$25 = 5 \otimes 5$	$7 \oplus 5^{\oplus 2} \oplus 3 \oplus 1$	7, 5^2 , 3^2 , 1^2
$35 = 5 \otimes 7$	$7 \oplus P(5) \oplus P(3)$	7, 5^3 , 3^4 , 1
$36 = 6 \otimes 6$	$7 \oplus P(5) \oplus P(3) \oplus 1$	7, 5^3 , 3^4 , 1^2
$49 = 7 \otimes 7$	$7^{\oplus 2} \oplus P(5) \oplus P(3) \oplus P(1)$	7^2 , 5^4 , 3^4 , 1^3

Having given restrictions of modules, we now need to understand Ext^1 between simple modules. These were completely determined in [3], but the information is not so easy to extract, and so we give a few special cases that are necessary for us.

Of particular interest is which modules have non-trivial 1-cohomology, since we will often want to prove that we stabilize a line. The next lemma gives this completely.

Lemma 5.24 Let p be a prime, $a \geq 1$ be an integer, and let M be a simple module for $H = \text{SL}_2(p^a)$ with non-trivial 1-cohomology. One of the following holds.

- (i) $p^a = 2$, M is the trivial module, with $\dim(H^1(H, M)) = 1$.

- (ii) p is odd and $a = 1$, $\dim(M) = p - 2$, with $\dim(H^1(H, M)) = 1$.
- (iii) $p^a = 9$, $\dim(M) = 4$, with $\dim(H^1(H, M)) = 2$.
- (iv) $p^a \neq 9$ with $a \geq 2$, M is up to application of a Frobenius map $L(p - 2) \otimes L(1)^\sigma$, where σ is the Frobenius twist (so that $\dim(M) = 2(p - 1)$ and $M = 2(p - 1)_{2,1}$), with $\dim(H^1(H, M)) = 1$.

Just knowing that modules have 1-cohomology is not going to be enough information. We need more specific information about extensions between simple modules of low dimension, for $p = 5, 7$ and $a > 1$. The final two lemmas of this section furnish us with this information.

Lemma 5.25 Let $H = \text{PSL}_2(5^a)$ for $a = 2, 3$. The extensions between simple modules of dimension at most 8 are:

- (i) 1 with $8_{i,i-1}$;
- (ii) 3_i with $4_{i,i+1}, 8_{i,i-1}$;
- (iii) $4_{i,i+1}$ with $3_i, 8_{i+1,i-1}$ (the latter only for $a = 3$);
- (iv) 5_i with nothing;
- (v) for $a = 2$, $8_{i,i+1}$ with 3_i ;
- (vi) for $a = 3$, $8_{i,i+1}$ with $4_{i-1,i}$;
- (vii) for $a = 3$, $8_{i,i-1}$ with $1, 3_i$.

Lemma 5.26 Let $H = \text{PSL}_2(49)$. The extensions between simple modules of dimension at most 9 are:

- (i) 1 with nothing;
- (ii) 3_i with $8_{i+1,i}$;
- (iii) $4_{1,2}$ with $5_1, 5_2$;
- (iv) 5_i with $4_{1,2}$;
- (v) 7_i with nothing;
- (vi) $8_{i,i+1}$ with $3_{i+1}, 9_{1,2}$;
- (vii) $9_{1,2}$ with $8_{1,2}, 8_{2,1}$.

6 Some PSL_2 s inside E_6 in characteristic 3

In this short section we lay the groundwork for studying copies of $H = \mathrm{PSL}_2(3^a)$ (for $a \geq 2$) inside $\hat{G} = F_4(k)$ by embedding \hat{G} inside $G = E_6(k)$, and attempting to construct many subgroups isomorphic to H inside members of \mathcal{X}^σ of G other than \hat{G} . Notice that field automorphisms of \hat{G} lift to field automorphisms of G , so we can embed an almost simple group with socle \hat{G} into an almost simple group with socle $G/Z(G)$. Let \bar{G} denote such an almost simple group, and note that V_{\min} is \bar{G} -stable.

Suppose that $N_{\bar{G}}(H)$ is contained inside both a σ -stable member X of \mathcal{X} and inside \hat{G} . The dimensions of G and \hat{G} are 78 and 52 respectively, so if X has dimension greater than 26, then $X \cap \hat{G}$ is positive dimensional (and of course still σ -stable), so that $N_{\bar{G}}(H)$ lies inside a σ -stable, positive-dimensional subgroup of \bar{G} , as desired. Since the Borel subgroup of G has dimension 42, if $N_{\bar{G}}(H)$ is contained in any parabolic subgroup of G then we are done.

This also works if $N_{\bar{G}}(H)$ is contained inside a conjugate of \hat{G} , inside C_4 – which has dimension 36 – and A_1A_5 – which has dimension 38. We are left with the irreducible G_2 , the A_2G_2 subgroup, and the $A_2A_2A_2$ maximal-rank subgroup. In these cases we will show that H is a blueprint for V_{\min} .

If H is contained in $X = G_2(k)$ then H acts on the natural module (up to Frobenius) as either $3_1^{\oplus 2} \oplus 1$, or lies in a diagonal subgroup of $A_1\tilde{A}_1$ acting as $4_{1,i} \oplus 3_1$ for any $i \geq 1$. In both cases H is contained in an algebraic A_1 subgroup Y .

The subgroup X acts irreducibly as $L(20)$ on V_{\min} , and these two copies of H act on V_{\min} as

$$3_1^{\oplus 3} \oplus (1/4_{1,2}/1)^{\oplus 3}, \quad 9_{1,i} \oplus 1/4_{1,2}/1 \oplus 4_{1,i}/(2_2 \otimes 2_i)/4_{1,i},$$

where of course $2_2 \otimes 2_i$ is $4_{2,i}$ if $i > 2$ and $1 \oplus 3_2$ if $i = 2$. The subgroups Y containing them act in the same way, and stabilize the same subspaces as H , so that H is a blueprint for V_{\min} .

If $X = A_2G_2$, then X acts on V_{\min} as $(10 \otimes 10) \oplus (02 \otimes 00)$: if H lies inside the G_2 factor then it centralizes a 6-space on V_{\min} , so definitely lies inside a line stabilizer of F_4 , and hence we may assume that H projects along the A_2 factor as 3_1 . Along the G_2 it can act as either $3_i^{\oplus 2} \oplus 1$ or $4_{i,j} \oplus 3_i$, for any $i, j \geq 1$ with $i \neq j$. In the first case we get

$$3_1 \oplus 9_{1,i}^{\oplus 2} \oplus 1/4_{1,2}/1 \quad \text{and} \quad 3_1^{\oplus 3} \oplus (1/4_{1,2}/1)^{\oplus 3},$$

for $i > 1$ and $i = 1$ respectively, and in the second case we get

$$3_1 \oplus 1/4_{1,2}/1 \oplus 4_{1,i}/(2_2 \otimes 2_i)/4_{1,i} \oplus 1/4_{1,2}/1, \quad 9_{1,i} \oplus 4_{1,j}/(2_2 \otimes 2_j)/4_{1,j} \oplus 1/4_{1,2}/1, \quad 9_{1,i} \oplus 12_{i,j,1} \oplus 1/4_{1,2}/1,$$

for $i = 1, j = 1$ and $i, j \neq 1$ respectively. Again, in all cases, the algebraic subgroup containing H stabilizes the same subspaces as H , so again H is a blueprint for V_{\min} .

Finally, we have $X = A_2A_2A_2$, which acts on V_{\min} as (up to duality) the sum of the three possible configurations of natural times natural times trivial. If we act trivially on one or two or them we get, up to twist,

$$3_1^{\oplus 6} \oplus 1^{\oplus 9}, \quad 3_1^{\oplus 7} \oplus 1/4_{1,2}/1, \quad 9_{1,i} \oplus 3_1^{\oplus 3} \oplus 3_i^{\oplus 3},$$

according as one non-trivial, two non-trivial and the same, and two non-trivial and different. Thus we may assume that H acts along the first factor as 3_1 , the second as 3_i and the third as 3_j , then we have, up to twist, one of

$$3_1^{\oplus 3} \oplus (1/4_{1,2}/1)^{\oplus 3}, \quad 3_1 \oplus 1/4_{1,2}/1 \oplus 9_{1,j}^{\oplus 2}, \quad 27_{1,i,j},$$

according as whether $i = j = 1$, $i = 1 \neq j$, and $1 \neq i \neq j \neq 1$. As with the other cases, each of these is contained in an algebraic A_1 stabilizing the same subspaces of V_{\min} , so again H is a blueprint for V_{\min} .

In particular, in all cases, either $N_{\bar{G}}(H)$ is contained in a member of \mathcal{X}^σ or H is a blueprint for the \bar{G} -stable module V_{\min} , in which case $N_{\bar{G}}(H)$ is contained in a member of \mathcal{X}^σ again.

We therefore have the following result.

Proposition 6.1 Let $H = \text{PSL}_2(3^a)$, and let \bar{G} be an almost simple group with socle $G = F_4(k)$. If H stabilizes a 3-space on the 25-dimensional minimal module for G , or the image of H in E_6 centralizes a 2-space on the 27-dimensional minimal module for E_6 , then $N_{\bar{G}}(H)$ is contained in a member of \mathcal{X}^σ

7 Proof of the theorems: strategy

In this section we discuss the techniques that we will use in proving that a given copy of $\mathrm{PSL}_2(p^a)$ is contained inside a member of \mathcal{X}^σ .

We have two cases to consider: when k , the field for the ambient group G , is not a splitting field for H , and when k is. In the first case, we have to be more careful, as for example if 2_1 is a submodule of $V_{\min} \downarrow_H$, H will not fix a 2-space over k , since 2_1 is not defined over k . Often we will deal with these small fields separately, so we can use more uniform arguments in general, but H fixes a line over k if and only if it does so over an extension field of k , so if that is what we prove we need no restriction on k .

The first step is usually to use the dimensions of modules and the traces of semisimple elements to produce a list of potential sets of composition factors for the action of H on V_{\min} , which we call *conspicuous* sets of composition factors. For many groups this list is small, but as the sizes of G and H grow the number grows larger and we need more efficient methods that cut this number down, for example only considering possible multisets of dimensions that have either no modules of dimension 1 or more modules of dimension $2(p-1)$ than modules of dimension 1, at least for p^a odd and not equal to 9, i.e., modules of positive pressure.

Having done this, we can assume we know the composition factors of $V_{\min} \downarrow_H$, and we have a few ways to proceed.

- We can use the traces of semisimple elements to deduce a list, often a list with one element, of potential sets of composition factors for $L(G) \downarrow_H$. Sometimes this cannot exist, of course only if there is no embedding of H with these composition factors. Other times $L(G) \downarrow_H$ has non-positive pressure, so we are again done. Otherwise, we may analyse both $V_{\min} \downarrow_H$ and $L(G) \downarrow_H$ using the techniques below. We will occasionally employ Lemma 2.1 in this regard.
- We can easily compute Ext^1 between the composition factors of $V_{\min} \downarrow_H$ and determine if $V_{\min} \downarrow_H$ is semisimple or not. If it is, the action of a unipotent element u must match one of the unipotent classes of G , whose actions on V_{\min} and $L(G)$ are tabulated in [13]. If it does not appear, or is generic, then we are done.
- If $V \downarrow_H$ is not semisimple, and V is self-dual (i.e., all cases except when $G = E_6$ and $V = V_{\min}$) then in order for a composition factor to appear in the socle and not be a summand, it must occur with multiplicity at least 2. This allows us to cut down the possibilities for the socle of $V \downarrow_H$.
- If the socle of $V \downarrow_H$ is W , then V is a submodule of $P(W)$, where $P(W)$ denotes the projective cover of W . In particular, it is a submodule of the $\mathrm{cf}(V)$ -radical of $P(W)$, where we recall that $\mathrm{cf}(V)$ is the set of composition factors of V . This needs to contain at least as many copies of each composition factor as there are in V , and further analysis of this radical can eliminate more cases.
- We can use Lemma 4.12: suppose that $H = \mathrm{PSL}_2(p^a)$ embeds in G , and an algebraic A_1 -subgroup X embeds in the algebraic version of G , such that for some module V , the highest weights of the composition factors of both H and X on V are the same. Assume furthermore that the composition factors of $V \downarrow_X$ satisfy the hypotheses of Lemma 4.12. We wish to conclude that H is a blueprint for V_{\min} . In order to do this, an element x in H of order $(p^a \pm 1)/2$ must be guaranteed to come from a class intersecting X . If the semisimple class containing x is determined by its eigenvalues on V then this is true, but this is not true for every semisimple class, so we will have to check when we use the lemma.

- In a similar vein, we can look for elements of G that do not lie in H and yet stabilize some eigenspaces of an element of H on a module V : if ζ_1, \dots, ζ_r are roots of unity and y acts as a scalar on each ζ_i -eigenspace of an element $x \in H$ (i.e., preserves all subspaces of the eigenspace), then y stabilizes any subspace of V on which x acts with eigenvalues some of the ζ_i . In particular, if there is a submodule W of H with this property then $\langle H, y \rangle$ stabilizes W . Of course, it might be that $\langle y, H \rangle$ is almost simple, say $\mathrm{PGL}_2(p^a)$ for example, so we need to exclude this case by finding other such elements, proving that the index of H in this group is not 2, or applying Corollary 4.14.
- If $G = E_6, E_7$ and $p = h - 1$ where h is the Coxeter number of G , then in one case we prove that H and $N_{\bar{G}}(H)$ stabilize an \mathfrak{sl}_2 -subalgebra of $L(G)$. We can then apply Corollary 4.19 on positive-dimensionality of such a stabilizer.

Some combination of these ideas is usually enough to solve any case we will see here.

8 F_4

In this section, k is a field of characteristic $p \geq 2$ and $G = F_4(k)$. Let \bar{G} be an almost simple group with socle G . From Section 4.2 we see that $v(F_4) = 18$, so if H is any subgroup of G with a semisimple element of order at least 19, then H is a blueprint for V_{\min} . The same holds for \bar{G} except possibly if $p = 2$ and \bar{G} induces a graph involution on G . In [10] we proved that, when $p = 2$, $\mathrm{SL}_2(2^a)$ cannot be a maximal subgroup of \bar{G} if $a \geq 5$ regardless of this potential problem with graph automorphisms. In addition, in [9] we proved that $\mathrm{SL}_2(4)$ cannot be a maximal subgroup of \bar{G} either, so here we let $H = \mathrm{PSL}_2(p^a)$ with $a = 3, 4$ if $p = 2$ and $p^a \leq 36 = 2 \cdot v(F_4)$ if p is odd. Let $L = \mathrm{PSL}_2(p) \leq H$ and let u denote a unipotent element of L of order p .

8.1 Characteristic 2

Let $p = 2$. Since all semisimple elements of G lie inside D_4 , which centralizes a 2-space on V_{\min} , and an element of order $2^a + 1$ in H has a fixed point only on the trivial simple module, we see that $V_{\min} \downarrow_H$ has at least two trivial composition factors. In particular, Lemma 5.2 applies in this situation, and so the pressure of $V_{\min} \downarrow_H$ has to be at least 2 for H not to fix a line on V_{\min} .

We start by eliminating the possibility that k is not a splitting field for H , or where the composition factors of $V_{\min} \downarrow_H$ are invariant under some outer automorphism of H , i.e., a field automorphism of H .

Proposition 8.1 Suppose that $p = 2$ and $a = 3, 4$, and suppose that k does not contain a splitting field for H , or that the composition factors of $V_{\min} \downarrow_H$ are invariant under a field automorphism of H . Then either H or its image under the graph automorphism fixes a line on V_{\min} .

Proof: Firstly let $a = 3$. The quickest way to do this is to use the traces of semisimple elements to see that the composition factors of $V_{\min} \downarrow_H$ must be

$$8^2, (2_1, 2_2, 2_3), 1^4, \quad (4_{1,2}, 4_{1,3}, 4_{2,3}), (2_1, 2_2, 2_3)^2, 1^2, \quad 8^3, 1^2.$$

The trace of an element of order 7 on these modules is 5, -2 and 5 respectively, and the trace of an element of order 9 is 2, -1 and -1 respectively. Using Lemma 4.16, we therefore see that the first and second sets of composition factors are swapped by the graph automorphism and the third does not exist. The first has negative pressure, so fixes a line on V_{\min} , as needed.

Now let $a = 4$. We assume that k contains \mathbb{F}_4 but not \mathbb{F}_{16} , as this includes the case where $k \cap \mathbb{F}_{16} = \mathbb{F}_2$. There are ten conspicuous sets of composition factors for $V_{\min} \downarrow_H$, two of which are definable over \mathbb{F}_2 . They fall into three orbits under the graph automorphism (recall that the graph automorphism squares to a field automorphism), and well-chosen representatives of the three orbits are

$$4_{1,3}, (2_1, 2_3)^4, 1^6, \quad (8_{1,2,4}, 8_{2,3,4}), 4_{1,3}, (2_1, 2_3), 1^2, \quad 4_{1,2}, 4_{2,3}, 4_{3,4}, 4_{1,4}, 4_{1,3}, 4_{2,4}, 1^2.$$

The pressures of these modules are 2, 0 and -2 respectively, so the second and third definitely fix lines on V_{\min} . For the first, if $4_{1,3}$ lies in the socle of $V_{\min} \downarrow_H$ then since V_{\min} is self-dual, it must be a summand, so we can assume that it is not; hence the socle consists of 2_1 s and 2_3 s, else H fixes a line on V_{\min} . The $\{1, 2_1, 2_3, 4_{1,3}\}$ -radical of $P(2_1)$ is

$$1/2_1, 2_3/1, 4_{1,3}/2_1,$$

so there must be at least three 2_i in the socle of $V_{\min} \downarrow_H$, but then $\mathrm{soc}(V_{\min} \downarrow_H)$ has pressure 3 but $V_{\min} \downarrow_H$ has pressure 2, contradicting Lemma 2.2. This completes the proof of the result. \square

We now turn to the case where k contains a splitting field for H . We split $a = 3$ and $a = 4$ into two propositions.

Proposition 8.2 Suppose that $p = 2$ and $a = 3$, and that k contains a splitting field for H . If $V_{\min} \downarrow_H$ or its image under the graph automorphism does not fix a line on V_{\min} , then up to field automorphism of H the composition factors of $V_{\min} \downarrow_H$ are

$$4_{1,3}^2, 2_1^4, 2_2, 2_3^2, 1^4,$$

H stabilizes a 2-space on V_{\min} , and $N_{\bar{G}}(H) = HC_{\bar{G}}(H)$ lies inside a member of \mathcal{X}^σ .

Proof: We use the traces of semisimple elements of order at most 17 to find all conspicuous sets of composition factors for $V_{\min} \downarrow_H$. There are sixty-three such sets, ten of which do not have corresponding sets of composition factors for $L(G) \downarrow_H$. The rest fall into orbits of length 2, 3 and 6 under the graph and field automorphisms of G . (The orbit of length 2 arose in Proposition 8.1, as did one of the ten sets that do not yield corresponding composition factors on $L(G)$.)

Note that there are two orbits of length six that share three points, because there are rational elements of order 9 whose image under the graph automorphism is non-rational, so there are three options for their image. This does not affect things, as the points they share correspond to conspicuous sets of composition factors with non-positive pressure so fix lines on V_{\min} .

Six of the ten orbits contain sets with pressure 0, and a further two contain sets of pressure 1, which are forbidden by Lemma 5.2. We are left with two orbits, one of length six and one of length three, with representatives

$$8, 4_{1,3}, 4_{2,3}, 2_1^2, 2_2, 2_3, 1^2, \quad \text{and} \quad 4_{1,3}^2, 2_1^4, 2_2, 2_3^2, 1^4.$$

Suppose that the first set of factors does not fix a line on V_{\min} : since the only non-trivial simple module appearing more than once is 2_1 , $V_{\min} \downarrow_H$ is the sum of an indecomposable module with socle 2_1 and a semisimple module, which we can ignore. We know that $V_{\min} \downarrow_H$ is self-dual, so if there is a non-split extension between two non-trivial simple modules A and B inside $V_{\min} \downarrow_H$, we must also have one between B and A : i.e., A (or B) must be 2_1 , and so the other must be $4_{1,3}$ by Lemma 5.1. In particular, $4_{2,3}$ must be a summand of $V_{\min} \downarrow_H$. The $\{1, 2_1, 2_2, 2_3, 4_{1,3}\}$ -radical of $P(2_1)$ has three trivial composition factors, but we remove all simple quotients other than 2_1 to obtain the smaller module

$$2_1/1/2_1, 2_2/1, 4_{1,3}/2_1,$$

on which an involution acts projectively. However, an involution cannot act projectively on V_{\min} (see [13, Table 3]) so H must fix a line on V_{\min} .

We come to the final set of composition factors. There are modules with these composition factors that do not fix a line, for example

$$4_{1,3}^{\oplus 2} \oplus 2_1/1/2_2/1/2_1 \oplus 2_1/1/2_3 \oplus 2_3/1/2_1,$$

and this has an allowable action of an involution as well. We note that $V_{\min} \downarrow_H$ always has a 2-dimensional submodule, and so lies inside a member of \mathcal{X}^σ , and furthermore, since $V_{\min} \downarrow_H$ is not stable under the field automorphism of H , if H is normalized by any element of an almost simple group \bar{G} with socle G then H is centralized by it.

To see that H stabilizes a 2-space on V_{\min} , note that otherwise $V_{\min} \downarrow_H$ is a submodule of $P(4_{1,3})$, but $P(4_{1,3})$ has structure

$$4_{1,3}/2_1/1/2_2/1/2_1/4_{1,3},$$

and there are many reasons why this cannot work: the dimension is 16, 2_3 is not involved in it, there are not enough 1s or 2_1 s, the involution u acts projectively on it, and so on. \square

There is a copy of $\mathrm{SL}_2(8)$ inside F_4 that does indeed not fix a line on V_{\min} up to the graph automorphism, inside $\tilde{A}_2 A_2$, with H projecting along the \tilde{A}_2 factor as $2_1/1$ and along the A_2 factor as $1/2_3$: the product of these two modules is an indecomposable module $2_1/1/2_3/4_{1,3}$ with dual $2_3/1, 4_{1,3}/2_1$, and the product of $2_1/1$ and its dual is

$$2_1/1/2_2/1/2_1 \oplus 1,$$

yielding an embedding into F_4 with the required property (remember that the trivial in the last decomposition is removed when considering V_{\min} .)

We now turn to $a = 4$, which ends this section on F_4 in characteristic 2. Almost exactly the same result holds in this case.

Proposition 8.3 Suppose that $p = 2$ and $a = 4$, and that k contains a splitting field for H . If $V_{\min} \downarrow_H$ or its image under the graph automorphism does not fix a line on V_{\min} , then up to field automorphism of H and graph automorphism of G the composition factors of $V_{\min} \downarrow_H$ are

$$4_{1,3}^2, 2_1^4, 2_2, 2_3^2, 1^4,$$

H stabilizes a 2-space on V_{\min} , and $N_{\bar{G}}(H) = HC_{\bar{G}}(H)$ lies inside a member of \mathcal{X}^σ .

Proof: We proceed as in Proposition 8.2, starting by assembling all conspicuous sets of composition factors using the traces of semisimple elements of order up to 17, this time finding 146 conspicuous sets of composition factors, 16 of which have no corresponding set of composition factors on $L(G)$, falling into eighteen orbits, fifteen of length 8, two of length 4 and one of length 2.

Eleven of these orbits contain a conspicuous set of composition factors with negative pressure, and a further two with factors with pressure 0. Using Lemma 5.2 we can exclude factors with pressure 1 as well, eliminating a further two orbits. There remain three orbits, each of length 8, so not stable under $\mathrm{Out}(H)$ or the graph automorphism of G .

The three orbits have representatives

$$8_{1,2,3}, 4_{1,3}, 4_{2,3}, 2_1^2, 2_2, 2_3, 1^2, \quad 8_{1,2,4}, 4_{1,3}, 4_{2,4}, 2_1^2, 2_2, 2_3, 1^2, \quad 4_{1,3}^2, 2_1^4, 2_2, 2_3^2, 1^4.$$

For the first orbit we argue as in Proposition 8.2, to see that since the only non-trivial simple module appearing more than once is 2_1 , so $V_{\min} \downarrow_H$ is a sum of a semisimple module, which can be ignored, and a self-dual submodule of $P(2_1)$ that has top 2_1 . The $\{1, 2_2, 2_3, 4_{1,3}, 4_{2,3}, 8_{1,2,3}\}$ -radical of $P(2_1)/2_1$, lifted back to $P(2_1)$, is

$$2_3/1, 4_{2,3}/2_2, 2_3/1, 4_{1,3}/2_1,$$

but on top of some submodule of this must go a 2_1 , and the module must be self-dual. We therefore see that we can remove all simple quotients other than 1 and $4_{1,3}$, as some submodule of this must be the second radical layer (as $1 \oplus 4_{1,3}$ is the second socle layer). Doing this yields the smaller module

$$1/2_2/1, 4_{1,3}/2_1,$$

and an involution u acts projectively on this, a contradiction.

The second orbit is almost identical, and the exact same proof works with the same smaller submodule of $P(2_1)$, and a slightly different original module of

$$1/2_2, 2_3/1, 4_{1,3}/2_1.$$

For the third orbit, since we constructed an example of this embedding not fixing a line on V_{\min} inside the $\tilde{A}_2 A_2$ subgroup just after Proposition 8.2, we will not be able to prove that it fixes a line on V_{\min} . However, it does stabilize a 2-space, as for $a = 3$: the $\{1, 2_1, 2_2, 2_3, 4_{1,3}\}$ -radical of $P(4_{1,3})$ is

$$2_2/1/2_1, 2_3/4_{1,3},$$

so $\text{soc}(V_{\min} \downarrow_H)$ cannot be just $4_{1,3}$, and H either fixes a line or a 2-space on V_{\min} , as needed. \square

8.2 Characteristic 3

In characteristic 3, since $v(F_4) = 18$ we only need consider $a = 2, 3$. We begin with the case $a = 3$, since we have a complete result for that. In fact, we show that $\text{PSL}_2(27)$ is always a blueprint for V_{\min} .

Proposition 8.4 Suppose that $p = 3$ and $a = 3$. Then H is a blueprint for V_{\min} .

Proof: There are forty conspicuous sets of composition factors for $V_{\min} \downarrow_H$, but for only seven of these do the elements of order 13 come from semisimple classes that are not blueprints for V_{\min} , and only three up to field automorphism of H , which are

$$12_{2,3,1}, 9_{1,2}, 4_{1,2}, \quad 12_{2,3,1}, 9_{1,3}, 4_{1,2}, \quad 4_{1,2}^2, 4_{1,3}^2, 4_{2,3}^2, 1.$$

Let ζ denote a primitive 13th root of unity, and let θ denote a primitive 26th root of unity with $\theta^2 = \zeta$. Let $x \in H$ denote an element of order 13 that acts on $4_{1,2}$ with eigenvalues $\zeta^{\pm 1}$ and $\zeta^{\pm 2}$. In the first case, x acts on V_{\min} with eigenvalues

$$1^2, (\zeta^{\pm 1})^2, (\zeta^{\pm 2})^3, (\zeta^{\pm 3})^2, (\zeta^{\pm 4})^2, \zeta^{\pm 5}, (\zeta^{\pm 6})^2.$$

There is an element \hat{x} of order 26 in G that squares to x and has eigenvalues

$$1^2, (\theta^{\pm 1})^2, (\theta^{\pm 2})^3, (\theta^{\pm 3})^2, (\theta^{\pm 4})^2, \theta^{\pm 5}, \theta^{\pm 6}, (-\theta^{\pm 6}).$$

This does not stabilize all the eigenspaces of x , but it only splits the $\zeta^{\pm 6}$ -eigenspaces, which are contained inside the 12 factor. Hence \hat{x} stabilizes all subspaces stabilized by H and, since \hat{x} has order $26 > v(F_4)$, this means that H is a blueprint for V_{\min} .

In the second and third cases, the trace of x on V_{\min} is 0, and there is an element \hat{x} in G , of order 26, such that $\hat{x}^2 = x$ and \hat{x} acts on V_{\min} with eigenvalues

$$1, (\theta^{\pm 1})^2, (\theta^{\pm 2})^2, (\theta^{\pm 3})^2, (\theta^{\pm 4})^2, \theta^{\pm 5}, (-\theta^{\pm 5}), \theta^{\pm 6}, (-\theta^{\pm 6}),$$

so \hat{x} stabilizes any $4_{1,2}$ in the socle of the second and third cases. This means that in the second case H is contained inside a positive-dimensional subgroup stabilizing the 4-space.

Examining the list of maximal positive-dimensional subgroups of G , if H acts on V_{\min} with factors 12, 9, 4 then the only member of \mathcal{X}^σ in which H can lie is $A_1 C_3$, which acts with factors of dimension 12 and 13, so any positive-dimensional subgroup containing H must also stabilize the 12; hence H is a blueprint for V_{\min} in this case as well.

We are left with the case of $4_{1,2}^2, 4_{1,3}^2, 4_{2,3}^2, 1$. We easily show that H lies inside a member of \mathcal{X} , (although not necessarily \mathcal{X}^σ yet). If H fixes a line on V_{\min} then we are done, so the socle consists of 4s, so by relabelling we get $4_{1,2}$ as a submodule of $V_{\min} \downarrow_H$. But now the element \hat{x} above must stabilize any $4_{1,2}$ in the socle of $V_{\min} \downarrow_H$, and since \hat{x} is a blueprint for V_{\min} , there is an infinite subgroup of G stabilizing a 4-space that $\langle H, \hat{x} \rangle$ stabilizes. Thus H is contained in an element of \mathcal{X} , as claimed.

We now run through the elements of \mathcal{X} , proving in fact that H does fix a line and u lies in the generic class A_2 , thus H is a blueprint for V_{\min} .

We cannot embed H in a maximal parabolic or $A_2\tilde{A}_2$ as the dimensions are not compatible. If $H \leq B_4$, which acts as $9 \oplus 16$, then by Lemma 5.12 H must act semisimply on the 9 (as the 9 is self-dual), so the action of u of order 3 has at least $3^6, 1^3$ on the 25-dimensional V_{\min} . Checking Table 4.1 we see that u belongs to class A_2 , generic.

If H embeds in $X = A_1C_3$ then we may assume that H acts along A_1 as 2_1 . Since $(1, 100)$ is a summand of $V_{\min} \downarrow_X$, we need 6-dimensional modules whose tensor product with 2_1 only have 4-dimensional composition factors, each appearing at most twice, and there are three of these: $2_2^{\oplus 2} \oplus 2_3$, $2_2 \oplus 2_3^{\oplus 2}$, and $6_{3,1}$, but none of these has the correct exterior square, so H does not embed in A_1C_3 .

We are left with A_1G_2 , which acts on V_{\min} with composition factors $(2, 10)$ of dimension 21 and $(4, 00)$ of dimension 4. This is impossible: H acts along the A_1 factor as 3_1 , and so the action of H on the minimal module for G_2 cannot have a trivial or 3-dimensional composition factor, because the product with 3_1 is not right. But then you cannot make a 7-dimensional module at all, a contradiction.

Thus if H embeds into G with these factors on V_{\min} then it is a blueprint, as needed. \square

We now consider $a = 2$, which we did not consider in [9] because we could not produce a complete answer there, and we cannot here either. We will make substantial progress, pinning down precisely the action of H on V_{\min} , but not enough to prove that it is always contained inside a positive-dimensional subgroup.

Proposition 8.5 Suppose that $p = 3$ and $a = 2$. One of the following holds:

- (i) H fixes a line on V_{\min} or $L(G)$;
- (ii) H stabilizes a unique 3-space on V_{\min} ;
- (iii) the action of H on the minimal module for E_6 has two trivial submodules;
- (iv) up to field automorphism of H , the action of H on V_{\min} is

$$9 \oplus 4/1, 3_1/4 \oplus 4,$$

where $4/1, 3_1/4 = 4 \otimes 3_2$.

If (i), (ii) or (iii) hold, then $N_{\bar{G}}(H)$ is contained in a member of \mathcal{X}^σ .

Proof: Using the traces of semisimple elements of orders 2, 4 and 5, one finds, up to field automorphism of H , eight conspicuous sets of composition factors, namely

$$\begin{aligned} 3_1^6, 1^7, & \quad 4^3, 3_1^3, 1^4, & \quad 4^4, 3_1, 3_2, 1^3, & \quad 9, 4^2, 3_1, 3_2, 1^2 \\ 9, 4^3, 3_1, 1, & \quad 4, 3_1^7, & \quad 4, 3_1^6, 3_2, & \quad 9^2, 4, 3_1. \end{aligned}$$

The first case is semisimple and u lies in the generic class A_2 . The sixth and seventh have the trace of an involution being -7 on V_{\min} , whence it has trace 20 on $L(G)$, so that $L = \text{PSL}_2(3)$ acts with composition

factors $3^8, 1^{28}$. By Lemma 5.8 any non-trivial simple module for H has at most two 3s for every three 1s on restriction to L , and so H always has at least sixteen trivial composition factors and at most eight non-trivial composition factors. Since $H^1(H, 4)$ has dimension 2 by Lemma 5.9, $L(G) \downarrow_H$ has non-positive pressure and hence has a trivial submodule. The second conspicuous set of composition factors must yield a trivial submodule on V_{\min} by Lemma 5.11. This leaves the third, fourth, fifth and eighth conspicuous sets of composition factors.

Suppose that we show that H stabilizes a unique 3-space on V_{\min} . The same must be true of $N_{\bar{G}}(H)$, and so if the stabilizer of that 3-space is positive-dimensional then $N_{\bar{G}}(H)$ lies inside a member of \mathcal{X}^σ , and in particular it is not maximal. However, Proposition 6.1 shows that if H stabilizes a 3-space on V_{\min} then the stabilizer of that 3-space is positive dimensional, and so we are done whenever this is the case. In particular, the semisimple eighth case must yield a unique 3-space being stabilized.

Also, if $V_{\min} \downarrow_H$ has no 1-cohomology then H stabilizes a line other than the F_4 line, and Proposition 6.1 again states that $N_{\bar{G}}(H)$ lies inside a member of \mathcal{X}^σ .

If the composition factors are $9, 4^2, 3_1, 3_2, 1^2$, and we fix neither a line nor a unique 3-space on V_{\min} , then the structure of $V_{\min} \downarrow_H$ is either

$$9 \oplus 4/1, 1, 3_1, 3_2/4 \quad \text{or} \quad 9 \oplus 3_1 \oplus 3_2 \oplus 4/1, 1/4.$$

We claim that, in either case, $V_{\min} \downarrow_H$ has no 1-cohomology. The second module is uniquely determined and so is an easy computer calculation. In the first case, the quotient by the unique 4 in the socle is also uniquely determined, and this module has no 1-cohomology; adding a 4 on the bottom that already has two trivial modules above it cannot add to the 1-cohomology, and so the claim holds. This proves, from the remarks above, that $N_{\bar{G}}(H)$ lies inside a member of \mathcal{X}^σ .

For $4^4, 3_1, 3_2, 1^3$, we cannot have a single 3 in the socle of $V_{\min} \downarrow_H$ as we saw above. We cannot have two 3s in the socle, for then the module has $3_1 \oplus 3_2$ as a summand and $4, 4/1, 1, 1/4, 4$, but since the self-dual module $4/1, 1/4$ is the $\{1, 4\}$ -radical of $P(4)$, such a module cannot exist.

Thus $V_{\min} \downarrow_H$ must have the form $4, 4/1, 1, 1, 3_1, 3_2/4, 4$. We now look at the image of H inside E_6 , and its action on the 27-dimensional module V_{27} . If $\text{soc}(V_{27} \downarrow_H) = 1$ then, since $P(1)$ has dimension 27, we have that $P(1) = V_{27} \downarrow_H$. However, the action of u of order 3 on V_{27} is clearly now 3^9 , so acts on V_{\min} as $3^8, 1$, from Table 4.1. But if we remove the top and socle from $P(1)$ we get a 25-dimensional module on which u acts as $3^7, 2^2$, a contradiction.

Thus there exists an H -submodule $1 \oplus 4$ of the minimal module for E_6 . Notice that $P(4)$ has dimension 36 and has five socle layers, and $P(1)$ has five socle layers, so since neither of these is contained in the module $V_{27} \downarrow_H$ (where V_{27} is the minimal module for E_6), we must have that $V_{27} \downarrow_H$ has at most four socle layers. In particular, since V_{27} is self-dual, we cannot have a uniserial module $3_i/4/1$ as a subquotient of $V_{27} \downarrow_H$, since $1/4/3_i$ would also have to be a subquotient, and there is a unique 3_i in $V_{27} \downarrow_H$, hence $V_{27} \downarrow_H$ needs at least five socle layers, not allowed.

Consider the preimage W of $\text{soc}^2(V_{\min} \downarrow_H)$ in V_{27} , and in particular the $\{1, 4\}$ -radical of W . This is the preimage of a module $1, 1/4 \oplus 1/4$, and since $1, 1/4$ has no extensions with 1, the $\{1, 4\}$ -radical of W must be a module $1, 1/4 \oplus 1/4/1$ (the uniserial module $1/4/1$ is not uniquely determined). We need to place both a 3_1 and a 3_2 on this module, but without constructing a uniserial $3_i/4/1$: this must yield a module

$$1, 1, 3_1, 3_2/4 \oplus 1/4/1,$$

the only way to exclude the uniserial $3_i/4/1$. Since there is no uniserial module $4/1/4$ by Lemma 5.12, no 4

placed on top of this module W can cover the 1, and so $V_{\min} \downarrow_H$ has a trivial quotient, not allowed. Thus (without loss of generality) 3_1 is diagonally placed across the $1, 1/4$ and the $1/4/1$.

Since $\text{Ext}^1((1, 1/4), 3_1) = 0$, at most one (and in fact exactly one) of the modules $1/4/1$ can have an extension with 3_1 , and by replacing the summands by diagonal summands if necessary, we end up with a module

$$1, 1/4 \oplus 1, 3_1/4/1,$$

and we have our uniserial $3_1/4/1$, not allowed. We therefore see that H cannot have non-trivial 1-cohomology, as needed for the proposition.

The final case to consider is $9, 4^3, 3_1, 1$. Again we cannot have the 3_1 in the socle, and not a trivial either, and so $V_{\min} \downarrow_H$ must be

$$9 \oplus 4/1, 3_1/4 \oplus 4.$$

There is a unique self-dual module $4/1, 3_1/4$, and it is $4 \otimes 3_2$. To see that it is unique, notice that above $1, 1, 3_1/4$ one may place two copies of 4, one of which lies above $1, 1$ only to make the module $4/1, 1/4$, and so any diagonal 4 between this and the one above $4 \otimes 3_2$ must cover both 1s, so we cannot peel a 1 off as a quotient to make a different $4/1, 3_1/4$. This proves (iv). \square

We can give a bit more information about case (iv) now. The action of such an H on V_{27} is unique as well: although $4/1, 3_1/4$ has non-trivial 1-cohomology, if the trivial does not lie only underneath the summand 4, there must be five socle layers to $V_{27} \downarrow_H$, which as we saw in the previous case leads to a contradiction.

This set of composition factors corresponds to $9, 4^4, 3_1^5, 3_2^3, 1^3$ on $L(G)$, but even this does not help as H can act on $L(G)$ without fixing a line. We cannot say much about the 3_i s, so remove them from the top and bottom, as well as the 9, to leave a module W . With four 4s and three 1s we cannot have $P(4)$ or $P(1)$, so from their structure above W has (at most) three socle layers. The two unipotent classes that act on V_{\min} as $3^8, 1$ are \tilde{A}_2 and $\tilde{A}_2 + A_1$: the former acts on $L(G)$ with seven blocks of size 1, and so we end up fixing a line, but the other acts as $3^{16}, 2^2$, and need not. With two 4s in the socle of W , and three 1s above, we get as in the previous analysis $1, 1/4 \oplus 1/4$, on which we can place 3_i s and then two 4s. We must place 3_1 or 3_2 on top of the $1/4$ to avoid fixing a hyperplane, and then on top of the $1, 1/4$ we need a 3_1 and a 3_2 , since we need both classes of elements of order 3 to act as $3^{16}, 2^2$ and modules of the form $4/1, 1/4$ and $4/1, 1, 3_i/4$ do not allow this. However there is, unique up to isomorphism, a self-dual indecomposable module $4/1, 1, 3_1, 3_2/4$ on which both unipotent classes act with two blocks of size 2.

Inside $C_3 A_1$, there is a copy of H whose projections along each factor act irreducibly on the respective natural modules, and it acts on V_{\min} as stated, and on $L(G)$ as

$$3_1^{\oplus 3} \oplus 9 \oplus 3_2/4/1, 3_1/4/3_2 \oplus 4/1, 1, 3_1, 3_2/4.$$

We cannot push the analysis far enough to get uniqueness of this subgroup: it certainly exists, as we have seen.

8.3 Characteristic at least 5

Let $p \geq 5$, and recall that $H = \text{PSL}_2(p^a)$ for some $a \geq 1$, with $p^a \leq 36 = 2 \cdot v(F_4)$, with $u \in H$ of order p . The possible actions of u on V_{\min} are given in [13, Table 3]; by Lemma 4.6 we may assume that our unipotent class is not generic: this leaves us with the following three unipotent classes:

- (i) $C_3, p = 7$, acting as $7^2, 6^2$;

- (ii) $F_4(a_2)$, $p = 7$, acting as $7^3, 5$;
- (iii) F_4 , $p = 13$, acting as 13^2 .

This proves the following result immediately.

Proposition 8.6 If $p^a \neq 7, 13$ then H is a blueprint for V_{\min} .

For $p = 7$ we have the following result.

Proposition 8.7 If $p^a = 7$ then H fixes a line on either V_{\min} or $L(G)$.

Proof: We use the traces of elements of orders 2, 3 and 4 to produce the possible composition factors of $V_{\min} \downarrow_H$, namely

$$3^6, 1^8, \quad 5, 3^7, \quad 5^3, 3^3, 1^2, \quad 7, 5^3, 3, 1, \quad 7^3, 1^5.$$

We saw in Section 5.3 that the only indecomposable module with the trivial composition factor but no trivial submodule or quotient is $P(3) = 5/1, 3/5$. This immediately tells us that the first, third and fifth cases fix lines on V_{\min} . (Indeed, the first and fifth cases have that all trivial factors are summands.)

The case $7, 5^3, 3, 1$ yields traces of elements of orders 2, 3 and 4 of 2, -1 and -2 respectively. This yields traces on $L(G)$ of -4 for an involution, -2 or 7 for an element of order 3, and finally 4 for an element of order 4. There is no set of composition factors that are compatible with this, so this case cannot occur.

If the composition factors are $5, 3^7$, then the traces of the elements of orders 2 and 3 fix uniquely the composition factors on $L(G)$, and these become $5^7, 3, 1^{14}$, and so H fixes lines on $L(G)$, completing the proof. \square

For $p = 13$ we are left with one open possibility, which we will prove yields a Serre embedding (see Definition 4.7).

Proposition 8.8 Suppose that $p^a = 13$. Either H is a blueprint for V_{\min} , or u is a regular unipotent element and $V_{\min} \downarrow_H$ and $L(G) \downarrow_H$ are given by

$$P(9) = 9/3, 5/9 \quad \text{and} \quad P(3) \oplus P(11) = 3/9, 11/3 \oplus 11/1, 3/11$$

respectively. In particular, H is a Serre embedding.

Proof: From the list above, the regular class is the only non-generic one for $p = 13$, so if H is not a blueprint for V_{\min} then u is regular and in particular acts projectively on V_{\min} and $L(G)$, hence both modules must restrict to H as projectives. The projective indecomposable modules for H are

$$1/11/1, \quad 3/9, 11/3, \quad 5/7, 9/5, \quad 7/5, 7/7, \quad 9/3, 5/9, \quad 11/1, 3/11, \quad 13.$$

Thus there are eight possible projective modules of dimension 26, two of which yield conspicuous sets of composition factors on V_{\min} , namely $P(5)$ and $P(9)$. The first of these does not have corresponding factors on $L(G)$, and the second has factors $11^3, 9, 3^3, 1$, which yield the projective module $P(3) \oplus P(11)$, as claimed. \square

We have therefore completed the proof of Theorem 1.1.

9 E_6

In this section, k is a field of characteristic $p \geq 2$ and $G = E_6(k)$, by which we mean the simply connected form, i.e., $|Z(G)| = \gcd(3, |k^\times|)$ (if k is finite) and $G' = G$. Let \bar{G} be an almost simple group with socle $G/Z(G)$. From [10] we see that for real semisimple elements (and the semisimple elements of $\mathrm{PSL}_2(p^a)$ are real), $v(E_6) = 18$, so if H is any subgroup of G with a real semisimple element of order at least 19, then H is a blueprint for V_{\min} . The same holds for \bar{G} , even when \bar{G} involves the graph automorphism, by using $V_{\min} \oplus V_{\min}^*$ instead of V_{\min} , which is \bar{G} -stable, and applying Lemma 4.9. In addition, in [9] we prove that almost simple groups with socles $\mathrm{SL}_2(4)$ and $\mathrm{PSL}_2(9)$ cannot be maximal subgroups of \bar{G} either, so here we let $H = \mathrm{PSL}_2(p^a)$ with $a = 3, 4$ if $p = 2$ and $p^a \leq 36 = 2 \cdot v(F_4)$ with $p^a \neq 9$ if p is odd. Let $L = \mathrm{PSL}_2(p) \leq H$ and let u denote a unipotent element of L of order p , as in Section 8.

9.1 Characteristic 2

Let $p = 2$. Unlike $G = F_4$ the case of $p = 2$ is easy, since the graph automorphism, which could cause the only problem, simply induces duality. As we see above, we just have to deal with $a = 3, 4$, and when $a = 3$ the group $\mathrm{Out}(H)$ has order 3, hence a graph automorphism must centralize H (hence H is an element of \mathcal{X}^σ), not merely normalize it. We therefore see that if $H = \mathrm{SL}_2(8)$ stabilizes a 2-space on V_{\min} then $N_{\bar{G}}(H)$ lies inside a positive-dimensional subgroup, even if \bar{G} induces a graph automorphism on G .

For $a = 4$ we use the fact that, while not every semisimple element of order 17 in F_4 is a blueprint for the minimal module, almost every one is. This statement passes through to V_{\min} , since our real semisimple elements lie in F_4 , via Lemma 4.9.

We start with $a = 3$.

Proposition 9.1 Suppose that $p = 2$ and $a = 3$. Then H fixes a line or 2-space on V_{\min} or V_{\min}^* .

Proof: Suppose that $\mathrm{soc}(V_{\min} \downarrow_H)$ and $\mathrm{soc}(V_{\min}^* \downarrow_H)$ have neither 1s nor 2s, so $V_{\min} \downarrow_H$ is a submodule of $P(4)$ s and 8s. The projective cover of $4_{i,i+1}$ is

$$4_{i,i+1}/2_{i+1}/1/2_{i-1}/1/2_{i+1}/4_{i,i+1},$$

and thus $V_{\min} \downarrow_H$ is a sum of projectives $P(4_{i,i+1})$ and 8s, but this has even dimension, not right. \square

Now we move on to $a = 4$, where we use semisimple elements of order 17 that are blueprints for V_{\min} , as suggested earlier.

Proposition 9.2 Suppose that $p = 2$ and $a = 4$. The subgroup H is always a blueprint for $V_{\min} \oplus V_{\min}^*$.

Proof: Of the 230 semisimple classes in F_4 of elements of order 17, all but two are blueprints for $V_{\min} \oplus V_{\min}^*$, as we saw in Section 8, with representatives x and x^3 , where x has eigenvalues

$$1^3, (\zeta_{17}^{\pm 1})^2, (\zeta_{17}^{\pm 2})^2, (\zeta_{17}^{\pm 3}), (\zeta_{17}^{\pm 4})^2, (\zeta_{17}^{\pm 5}), (\zeta_{17}^{\pm 6}), (\zeta_{17}^{\pm 7}), (\zeta_{17}^{\pm 8})^2$$

on V_{\min} . We thus may assume that every element of H of order 17 is conjugate to either x or x^3 .

However, although there are 107766 possible sets of composition factors for a module of dimension 27, none of them has the eigenvalues above, up to algebraic conjugacy. Thus a semisimple element of H of order 17 is always a blueprint for $V_{\min} \oplus V_{\min}^*$, and so the result holds by Lemma 4.9. \square

9.2 Characteristic 3

Let $p = 3$. From the remarks at the start of this section we need only consider $a = 3$, i.e., $H = \text{PSL}_2(27)$. In the previous section we exploited the fact that most semisimple elements of order 17 are blueprints for V_{\min} . We will do the same here with order 13 elements. Of the 104 semisimple classes of elements of order 13 in F_4 , all but seven are blueprints, since there are elements of order 26 that square to them and preserve the number of eigenvalues.

Proposition 9.3 Suppose that $p = 3$ and $a = 3$. Either H is a blueprint for $V_{\min} \oplus V_{\min}^*$ or it fixes a line on V_{\min} or V_{\min}^* .

Proof: This is easier than the case of F_4 , but will start in exactly the same way. There are fifty conspicuous sets of composition factors for $V_{\min} \downarrow_H$, but for only seven of these do the elements of order 13 come from semisimple classes that are not blueprints for $V_{\min} \oplus V_{\min}^*$, three up to field automorphism of H , which are

$$12_{2,3,1}, 9_{1,2}, 4_{1,2}, 1^2, \quad 12_{2,3,1}, 9_{1,3}, 4_{1,2}, 1^2, \quad 4_{1,2}^2, 4_{1,3}^2, 4_{2,3}^2, 1^3.$$

The first two have pressure -1 and so must fix a line on V_{\min} . For the third, since there is no module $4/1/4$ for H by Lemma 5.12, $V_{\min} \downarrow_H$ cannot have the form $4, 4, 4/1, 1, 1/4, 4, 4$, and so if H does not fix a line or hyperplane on V_{\min} then there can be either one or two 4s in the socle.

The $\{1, 4\}$ -radical of $P(4_{1,2})$ has three trivial factors but has a trivial quotient, so we may assume that the socle of $V_{\min} \downarrow_H$ is the sum of two 4s. Since the pressure of $V_{\min} \downarrow_H$ is 3, we cannot have a submodule with three 4s, but we need two 1s above this socle (else we could quotient out by one of them and get a module with a simple socle), so the socle is $4, 4$ and the second socle layer is $1, 1$, so we must have a $4/1/4$ subquotient, not allowed. Thus H fixes a line on V_{\min} or V_{\min}^* , as needed. \square

9.3 Characteristic at least 5

Let $p \geq 5$, and recall that $H = \text{PSL}_2(p^a)$ for some $a \geq 1$, with $p^a \leq 36$, with $u \in L \leq H$ of order p , where $L = \text{PSL}_2(p)$. The possible actions of u on V_{\min} are given in [13, Table 5]; by Lemma 4.6 we may assume that our unipotent class is not generic, leaving us with the following seven unipotent classes:

- (i) A_4 , $p = 5$, acting as $5^5, 1^2$;
- (ii) $A_4 + A_1$, $p = 5$, acting as $5^5, 2$;
- (iii) A_5 , $p = 7$, acting as $7^2, 6^2, 1$;
- (iv) $D_5(a_1)$, $p = 7$, acting as $7^3, 3, 2, 1$;
- (v) $E_6(a_3)$, $p = 7$, acting as $7^3, 5, 1$;
- (vi) $E_6(a_1)$, $p = 11$, acting as $11^2, 5$;
- (vii) E_6 , $p = 13$, acting as $13^2, 1$.

We now go prime by prime, starting with $p = 5$.

Proposition 9.4 Suppose that $p = 5$. If $a = 1$ then H fixes a line on either V_{\min} or $L(G)$. If $a = 2$ then H either fixes a line or hyperplane on V_{\min} , or a line on $L(G)$.

Proof: Suppose that $a = 1$. The conspicuous sets of composition factors of $V_{\min} \downarrow_H$ are

$$3^6, 1^9, \quad 5, 3^7, 1, \quad 5^3, 3^3, 1^3.$$

The first set of composition factors has pressure -3 , so fixes a line on V_{\min} by Lemma 2.2. In the second case we switch to $L(G)$, on which H acts with composition factors

$$5^8, 3^8, 1^{14} \quad \text{or} \quad 5^{11}, 3^5, 1^8.$$

In either case, we see that H fixes a line on $L(G)$, as needed. The third set of composition factors has pressure 0, so might only fix a hyperplane on V_{\min} . However, the only indecomposable modules with a trivial composition factor but no trivial submodule are submodules of $P(3) = 3/1, 3/3$, so in order not to fix a line, $V_{\min} \downarrow_H$ must be

$$5^{\oplus 3} \oplus (1/3)^{\oplus 3},$$

on which u acts as $5^3, 4^3$, but this does not appear on [13, Table 5], so H does indeed fix a line (and hyperplane) on V_{\min} .

Now suppose that $a = 2$. By Lemma 5.21, if $V_{\min} \downarrow_L$ has more trivials than 3-dimensionals then H fixes a line on V_{\min} . Thus if $V_{\min} \downarrow_L$ is the first set of composition factors then H fixes a line on V_{\min} , and if $V_{\min} \downarrow_L$ is the second set of composition factors then H fixes a line on $L(G)$.

We therefore assume that $V_{\min} \downarrow_L$ has factors $5^3, 3^3, 1^3$. At this point it seems easiest to use the traces of semisimple elements of order at most 13, finding eighteen conspicuous sets of composition factors, each with at least one trivial factor and with non-positive pressure, so fix either a line or a hyperplane on V_{\min} . \square

For $p = 7$ we do not need to go past $a = 1$, which makes this easier than the previous case.

Proposition 9.5 Suppose that $p = 7$ and $a = 1$. Then H fixes a line or hyperplane on V_{\min} .

Proof: The conspicuous sets of composition factors are, as for $p = 5$, the same as for F_4 but with an extra trivial factor, namely

$$3^6, 1^9, \quad 5, 3^7, 1, \quad 5^3, 3^3, 1^3, \quad 7, 5^3, 3, 1^2, \quad 7^3, 1^6.$$

The only indecomposable module that has a trivial composition factor but no trivial submodule or quotient is $P(5) = 5/1, 3/5$, thus all of these sets of composition factors fix either a line or hyperplane on V_{\min} . \square

Since any indecomposable module with a trivial composition factor but no trivial submodule has $5/1$ as a submodule, the first and fifth conspicuous sets of composition factors in the proof definitely fix lines on V_{\min} . If H fixes a hyperplane but not a line then it cannot lie in F_4 and must lie inside a D_5 -parabolic, with composition factors of dimensions 1, 16 and 10. These are incompatible with the second and fifth sets of composition factors, hence H also lies inside F_4 in this case.

For $p = 11$, we see the first use of the idea of fixing an \mathfrak{sl}_2 -subalgebra.

Proposition 9.6 Suppose that $p = 11$. Either H is a blueprint for both V_{\min} and $L(G)$, or H has a trivial summand on V_{\min} , or H acts on V_{\min} and $L(G)$ as

$$P(9) \oplus 5 \quad \text{and} \quad 11^{\oplus 2} \oplus P(7) \oplus P(5) \oplus 9 \oplus 3,$$

and fixes an \mathfrak{sl}_2 -subalgebra of $L(G)$.

Proof: Examining [13, Tables 5 and 6], we see that there are only two unipotent classes of elements of order 11 that are not generic for both V_{\min} and $L(G)$, namely D_5 (generic for V_{\min}) and $E_6(a_1)$ (not generic for either). If u belongs to class D_5 , then it acts on V_{\min} with Jordan blocks $11, 9, 5, 1^2$, and since there are two Jordan blocks of size 1 and only one of size 11, $V_{\min} \downarrow_H$ must have a trivial summand as each non-trivial indecomposable summand of dimension congruent to 1 modulo 11 has dimension 12 and uses up a block of size 11.

We therefore assume that u belongs to class $E_6(a_1)$, so acts as $11^2, 5$ on V_{\min} and as $11^6, 9, 3$ on $L(G)$. There are five indecomposable modules of dimension congruent to 5 modulo 11, which up to duality are

$$5, \quad 7, 5, 3/5, 3, \quad 9, 7, 5, 3/1, 3, 5, 7, 9,$$

with the last one having dimension 49, not allowed, and the second one having dimension 27, with trace of an involution -1 , so not allowed. Thus $V_{\min} \downarrow_H$ is the sum of 5 and a 22-dimensional projective module.

We now use traces of semisimple elements of orders at most 6 to see which sums of projectives and a 5 are conspicuous, finding two, namely

$$11 \oplus P(1) \oplus 5 \quad \text{and} \quad P(9) \oplus 5.$$

The first fixes a line on V_{\min} but does not have a trivial summand, hence lies inside a D_5 -parabolic, acting on V_{\min} uniserially as $10/16/1$, and the image of H inside the D_5 -Levi must act as $1/9$ on the 10, not allowed since this is a self-dual module. Thus the first case does not exist, and H must be the second.

The corresponding sets of composition factors on $L(G)$ are

$$11, 9^3, 7^4, 3^3, 1^3 \quad \text{and} \quad 11^2, 9, 7^3, 5^4, 3^2.$$

Since $L(G)$ is self-dual and there is a unique self-dual module congruent to each dimension modulo 11 by Lemma 5.17, we have that 9 and 3 must be summands of $L(G) \downarrow_H$. The first set of factors cannot form a projective and these summands, but the second case can, yielding

$$11^{\oplus 2} \oplus P(7) \oplus P(5) \oplus 9 \oplus 3.$$

By Proposition 4.17, the 3-dimensional summand is an \mathfrak{sl}_2 -subalgebra, as claimed. \square

When $p = 13$, the only non-generic class is the regular unipotent class. We will show more generally that if H contains a regular unipotent element then H either lies in F_4 , or $p = 13$ and H is a non- G -cr subgroup in a D_5 -parabolic subgroup of G .

Proposition 9.7 Suppose that $p \geq 13$. If H contains a regular unipotent element then H is contained in a conjugate of F_4 , or $p = 13$ and H is a non- G -cr subgroup of the D_5 -parabolic acting on V_{\min} as

$$1/11/1 \oplus 9/5.$$

or its dual.

If H does not contain a regular unipotent element, then H is a blueprint for $V_{\min} \oplus V_{\min}^*$.

Proof: Suppose that $p \geq 17$: the action of a unipotent element on V_{\min} is $17, 9, 1$, and for $p \geq 19$ we must have that $V_{\min} \downarrow_H = 17 \oplus 9 \oplus 1$, and so H lies inside either F_4 , as desired, or a D_5 -parabolic, but this has composition factors $10, 16, 1$, incompatible. For $p = 17$ then $17, 1$ could come from an 18-dimensional indecomposable module, but the 9 is a summand, so in particular H has three composition factors on V_{\min} .

However, u is contained in the regular class, which is generic for $p = 17$, hence H is a blueprint for V_{\min} , in particular an element X of \mathcal{X} . Since X contains a regular unipotent element (eliminating all reductive maximal subgroups except for F_4 from [14]) and must have at most three composition factors on V_{\min} , and if it does have three then one has dimension 9 (eliminating all parabolic subgroups), we must have $H \leq F_4$, as claimed.

We therefore have that $p = 13$, and u acts on V_{\min} with factors $13^2, 1$. Suppose that the 1 in the action of u arises from a trivial summand in $V_{\min} \downarrow_H$. From the proof of Proposition 8.8 we see that the conspicuous sets of composition factors are

$$5/7, 9/5 \oplus 1 \quad \text{and} \quad 9/3, 5/9 \oplus 1.$$

Since there is no 10-dimensional quotient not including the trivial summand, these structures are incompatible with coming from a D_5 -parabolic, and so $H \leq F_4$, as needed.

We thus assume that $V_{\min} \downarrow_H$ has no trivial summand. We therefore have a projective of dimension 13 (either $P(1)$ or 13, both with a trace of 1 for the involution) and a module $i/(p+1-i)$, with a trace of ± 2 . As the trace of an involution on V_{\min} is either 3 or -5 , we see that it has a trace of $+2$ on $i/(p+1-i)$, and hence $i = 5, 9$. This means that, up to duality, $V_{\min} \downarrow_H$ is either

$$13 \oplus 5/9 \quad \text{or} \quad 1/11/1 \oplus 9/5.$$

The second case is as claimed in the proposition, so we are left to eliminate the first case. Here we take the Borel subgroup B of H : the exact structure of B on the 27-dimensional module V_{\min} is up to duality as follows, where ζ is a cube root of unity.

$$\begin{array}{ccc} 1 & \zeta & \\ -\zeta & -\zeta^2 & \\ \zeta^2 & 1 & \\ -1 & -\zeta & \\ \zeta & \zeta^2 & \\ -\zeta^2 & -1 & \\ 1 & \zeta & \zeta^2 \\ -\zeta & -\zeta^2 & \\ \zeta^2 & 1 & \\ -1 & -\zeta & \\ \zeta & \zeta^2 & \\ -\zeta^2 & -1 & \\ 1 & \zeta & \end{array}$$

Since F_4 acts on V_{\min} as $26 \oplus 1$, the point that H fixes is either a D_5 -parabolic point or a B_4 point, but either way H lies inside a D_5 -parabolic, either one stabilizing a line or one stabilizing a hyperplane.

Let v be a unipotent element of D_5 contained in the image of L inside the D_5 -Levi. Thus v acts on the 10 and 16 as subquotients of the action of u on V_{\min} , namely $13^2, 1$. Therefore v acts on both the 10 and the 16 with at most three Jordan blocks, and if it has three then one is of size 1.

We can read off the unipotent classes of D_5 from the table for D_6 , [14, Table 6], which shows that there are only three unipotent classes, A_4 , $D_5(a_1)$ and D_5 , that act with at most three blocks on the 10. From the embedding of the D_5 -Levi into E_6 we can easily deduce the actions of these on the 16, as we just consult [13, Table 5] which lists the block sizes for the classes for E_6 , and look for the unipotent classes with these names. This gives us the list below.

Class	A_4	$D_5(a_1)$	D_5
Action on 10	5^2	$7, 3$	$9, 1$
Action on 16	$7, 5, 3, 1$	$7^2, 2$	$9, 7$

We therefore see that v comes from the regular class D_5 , and so the image \bar{B} of B in D_5 , which contains v , must act on the self-dual module 10 as

$$1 \oplus (\zeta^2 / -1/\zeta / -\zeta^2 / 1 / -\zeta / \zeta^2 / -1/\zeta).$$

This is a submodule of the action of B above, and we therefore see that \bar{B} acts on the 16 with eigenvalues

$$(1, -1)^2, (\zeta, \zeta^2, -\zeta, -\zeta^2)^3;$$

these cannot form modules of dimension 9 and 7, since a module of dimension 9 needs exactly three ± 1 eigenvalues, and a module of dimension 7 needs at least two ± 1 s.

This proves that H cannot embed with these composition factors, and completes the proof of the proposition. \square

We will construct this non- G -cr subgroup of the D_5 -parabolic when $p = 13$; the same construction works for the E_6 -parabolic subgroup of E_7 and $p = 19$.

Let (G, p, X, Y) be one of $(E_6, 13, D_5, B_4)$ and $(E_7, 19, E_6, F_4)$, and let V_{\min} denote the minimal module for G . One of the stabilizers of a point on V_{\min} is a subgroup that is the extension of a unipotent group by Y , so let H be a copy of $\mathrm{PSL}_2(p)$ inside Y that covers the regular unipotent element, the fixed points of a principal PSL_2 subgroup of Y . This copy of H embeds in X , of course, and the action of X on the unipotent radical of the X -parabolic is as a single simple module, so that the 1-cohomology is easy to compute. We see that the restriction of this simple module to H contains a summand of dimension $p - 2$, hence the 1-cohomology of H on the unipotent radical is 1-dimensional. There is an action of the torus of the X -parabolic outside of X on this cohomology group, and this yields two conjugacy classes of subgroups H in the X -parabolic, one inside X and another class of complements. Given the composition factors of H on V_{\min} , together with the table from [13], there is a unique possible module structure for $V_{\min} \downarrow_H$ if H does not lie inside X but merely the X -parabolic subgroup of G , and this intersects non-trivially the regular unipotent class of G .

10 E_7 in characteristic 2

In this section, let k be a field of characteristic 2 and let $G = E_7(k)$. Let $a \geq 1$ be a positive integer and $H = \mathrm{SL}_2(2^a)$. The case of characteristic 2 is very different from odd characteristic because if p is odd then a copy of $\mathrm{PSL}_2(p^a)$ inside the simple group of type E_7 can lift in the simply connected group to either $\mathrm{PSL}_2(p^a) \times 2$ or $\mathrm{SL}_2(p^a)$, and the two possibilities require very different strategies. In characteristic 2 there is no such bifurcation.

The case $a = 2$ is done in [9], and since $v(E_7) = 75$ for semisimple elements of odd order, if $a \geq 7$ then H is a blueprint for V_{\min} ; so we may assume that $3 \leq a \leq 6$. Furthermore, if $V_{\min} \downarrow_H$ has at least six trivial composition factors then by Proposition 4.10 we can assume that H has no semisimple elements of order more than 30, so $a \leq 4$ in this case.

We can use a computer to find which semisimple elements are blueprints for V_{\min} even when they have order smaller than 77, or 30 when they centralize a 6-space. For example, of the 2430 classes of elements of order 17, 1892 of them are blueprints for V_{\min} , which helps reduce the number of conspicuous sets of composition factors that need to be considered when $a = 4$.

Suppose firstly that there are no 1- or 2-dimensional composition factors in $V_{\min} \downarrow_H$. In this case if H is not a blueprint for V_{\min} then we have to switch to the Lie algebra $L(G)'$, which we recall has dimension 132, not 133 in the case $p = 2$. We address this situation now.

Proposition 10.1 Suppose that $p = 2$ and $a = 3, 4, 5, 6$. Suppose that there are no 1- or 2-dimensional composition factors in $V_{\min} \downarrow_H$.

- (i) We cannot have $a = 3, 4$.
- (ii) If $a = 5, 6$ then H is a blueprint for V_{\min} .

Proof: If H acts on V_{\min} with no composition factors of dimension 1 or 2, then the trace of an element of order 3 on V_{\min} is one of $-25, -7, 2, 20$, and so the dimensions of the composition factors are one of seven possibilities:

$$32, 16, 4^2, \quad 32, 8, 4^4, \quad 16^3, 8, \quad 16^2, 8^2, 4^2, \quad 16, 8^3, 4^4, \quad 8^7, \quad 8^4, 4^6.$$

For these, no arguments about unipotent classes or stabilizing subspaces will work if H is not a blueprint for V_{\min} , and so we will just have to deal with them case by case, switching to the Lie algebra where we need to.

Let $a = 3$: the only conspicuous set of composition factors for $V_{\min} \downarrow_H$ is

$$8^4, 4_{1,2}^2, 4_{1,3}^2, 4_{2,3}^2,$$

which does not have a corresponding set of composition factors on $L(G)'$. For $a = 4$ we get no conspicuous sets of composition factors for $V_{\min} \downarrow_H$ at all.

For $a = 5$, up to field automorphism of H , there are two conspicuous sets of composition factors for $V_{\min} \downarrow_H$, namely

$$16_{1234}, 8_{135}, 8_{235}, 8_{345}, 4_{13}^2, 4_{23}, 4_{34} \quad \text{and} \quad 8_{123}^2, 8_{124}, 8_{135}, 4_{12}, 4_{13}, 4_{14}^2, 4_{15}^2.$$

It is easy to check that an element of order 31 is a blueprint for V_{\min} in both cases by finding an element of order 93 that cubes to it and has the same number of eigenvalues on V_{\min} .

For $a = 6$, we do not have lists of the traces of elements of orders 63 and 65, but we can check whether a given matrix is the trace of a semisimple element of order 63 on V_{\min} by using the preimage trick from Section 4.2. Doing this to the seven possible sets of dimensions yields the following table. In this, the number of sets of composition factors up to field automorphism is given in the second column, and those that are conspicuous using elements of order up to 21 and 63 are given in the third and fourth columns respectively.

Dimensions	Number of modules	Conspicuous up to 21	Conspicuous for 63
$32, 16, 4^2$	1800	1	0
$32, 8, 4^4$	61200	5	0
$16^3, 8$	2270	1	0
$16^2, 8^2, 4^2$	504240	32	0
$16, 8^3, 4^4$	11781000	159	2
8^7	109660	1	0
$8^4, 4^6$	57206136	934	9

We thus simply need to check whether for a given conspicuous sets of composition factors, that any conjugacy class of elements of order 63 with the correct eigenvalues on V_{\min} is a blueprint for V_{\min} . This can easily be done with a computer, and so we prove the result. \square

We have now dealt with the case where $V_{\min} \downarrow_H$ has no 1- or 2-dimensional composition factors. We generally cannot prove that H fixes a line on V_{\min} , and often want to prove that H fixes a 2-space on V_{\min} . For this to be enough to show that H is contained in a positive-dimensional subgroup, we need that k contains a splitting field for H , and for $N_{\bar{G}}(H)$ we need to also consider when $V_{\min} \downarrow_H$ is stable under a field automorphism of H . In general, we therefore consider conspicuous sets of composition factors that are stable under some field automorphism.

Proposition 10.2 Suppose that $3 \leq a \leq 6$ and that the composition factors of $V_{\min} \downarrow_H$ are stable under a non-trivial field automorphism of H .

- (i) If $a = 3$ then either H fixes a line on V_{\min} or $L(G)'$, or the composition factors of $V_{\min} \downarrow_H$ are

$$8, (4_{1,2}, 4_{1,3}, 4_{2,3})^2, (2_1, 2_2, 2_3)^3, 1^6.$$

- (ii) If $a = 4$ then H is a blueprint for V_{\min} or the composition factors of $V_{\min} \downarrow_H$ are

$$(4_{1,4}, 4_{2,3})^2, 4_{1,2}^2, (2_1, 2_3)^4, (2_2, 2_4)^2, 1^8$$

and the stabilizer of any simple submodule of $V_{\min} \downarrow_{N_{\bar{G}}(H)}$ is positive dimensional.

- (iii) If $a \geq 5$ then H is a blueprint for V_{\min} .

Proof: Let $a = 3$. Over \mathbb{F}_2 there are eight conspicuous sets of composition factors for $V_{\min} \downarrow_H$, two of which have no corresponding set of composition factors on $L(G)'$. One is as in the proposition, with corresponding factors

$$8^2, (4_{1,2}, 4_{1,3}, 4_{2,3})^4, (2_1, 2_2, 2_3)^9, 1^{14}$$

on $L(G)'$.

The other five are

$$8, (4_{1,2}, 4_{1,3}, 4_{2,3})^3, (2_1, 2_2, 2_3)^2, (4_{1,2}, 4_{1,3}, 4_{2,3})^2, (2_1, 2_2, 2_3)^4, 1^8 \quad 8^6, 1^8,$$

$$8^4, (2_1, 2_2, 2_3)^2, 1^{12}, \quad 8, (2_1, 2_2, 2_3)^8,$$

with corresponding sets of factors on $L(G)'$ given by

$$\begin{aligned} &8^2, (4_{1,2}, 4_{1,3}, 4_{2,3})^2, (2_1, 2_2, 2_3)^{11}, 1^{26}, \quad 8^2, (4_{1,2}, 4_{1,3}, 4_{2,3})^4, (2_1, 2_2, 2_3)^9, 1^{14}, \\ &8^{11}, (4_{1,2}, 4_{1,3}, 4_{2,3}), (2_1, 2_2, 2_3)^3, 1^{14}, \quad 8^8, (4_{1,2}, 4_{1,3}, 4_{2,3}), (2_1, 2_2, 2_3)^6, 1^{20}, \\ &(4_{1,2}, 4_{1,3}, 4_{2,3})^8, (2_1, 2_2, 2_3), 1^{30}. \end{aligned}$$

By Lemma 5.4 we need three 2s for every two 1s in order not to fix a line or hyperplane, and so the third and fourth cases fix lines on V_{\min} and the first, third, fourth and fifth all fix lines on $L(G)'$, so we need to consider the second case. We assume that $V_{\min} \downarrow_H$ has no trivial submodules.

Remove any 4s from the top and bottom of $V_{\min} \downarrow_H$, and any summands of dimension 2 to yield a module W . Since $V_{\min} \downarrow_H$ has pressure 4, there are at most four 2s in the socle of W . We cannot have a summand $P(2_i)$ for any i because $P(2_i)$ has dimension 56 and only four trivial factors, so as in the proof of Lemma 5.4 we see that W has at most five socle layers, and in fact exactly five socle layers. As it has pressure 4 we cannot have a subquotient of pressure greater than 4 or less than -4 by Lemma 2.2, so the first, third and fifth socle layers each have four 2s, and the second and fourth socle layers each have four 1s.

It is clear that, since there are four 2s in the socle, two of them must be isomorphic, say 2_1 . The other two 2_1 s must be in the top, and so the socle is either $M_1 = 2_1^{\oplus 2} \oplus 2_2^{\oplus 2}$ or $M_2 = 2_1^{\oplus 2} \oplus 2_2 \oplus 2_3$. In the first case, we construct the $\{2_1, 2_2\}'$ -submodule of $P(M_1)/M_1$, and note that this has only six trivial composition factors, so it cannot be the right socle.

In the other case we construct the $\{2_1\}'$ -submodule of $P(M_2)/M_2$, which has exactly eight trivial composition factors, so they must all lie in W . Therefore we take the $\{1\}'$ -residual of this, and it has structure

$$1, 1, 1, 1/2_2, 2_2, 2_2, 2_3, 2_3/1, 1, 1, 1, 4_{2,3}/2_1, 2_1, 2_2, 2_3.$$

This must be a submodule of W , and yet it has five 2s in the third socle layer, too many. This contradiction proves that H fixes a line on V_{\min} , as needed.

Thus, except for the set of composition factors explicitly stated, H must fix a line on either V_{\min} or $L(G)'$.

For $a = 4$ there are, up to field automorphism of H , six conspicuous sets of composition factors over $k = \mathbb{F}_4$, namely

$$\begin{aligned} &4_{1,3}^2, (2_1, 2_3)^8, 1^{16}, \quad 4_{2,4}^8, (2_1, 2_3)^2, 1^{16}, \quad (8_{1,2,4}, 8_{2,3,4})^2, 4_{1,3}^2, (2_1, 2_3)^2, 1^8, \\ &(4_{1,4}, 4_{2,3})^2, 4_{1,2}^2, (2_1, 2_3)^4, (2_2, 2_4)^2, 1^8 \quad (4_{1,2}, 4_{1,3}, 4_{1,4}, 4_{2,3}, 4_{2,4}, 4_{3,4})^2, 1^8, \quad 16^2, (2_1, 2_2, 2_3, 2_4)^2, 1^8. \end{aligned}$$

In the proof of Proposition 9.2 we saw that there are only two semisimple classes of elements of order 17 in F_4 that are not blueprints for the 26-dimensional minimal module, and hence for V_{\min} . The semisimple elements of order 17 in the conspicuous sets of composition factors above are always conjugate into F_4 , and all but the fourth one are in fact blueprints for V_{\min} (as they are for the minimal module for F_4), so we get the first part of the result.

We are left with the fourth set of composition factors. Let ζ denote a primitive 17th root of unity, and choose $x \in H$ of order 17 such that the eigenvalues of x on 2_1 are $\pm\zeta$, and therefore the eigenvalues of x on V_{\min} are

$$1^8, (\zeta^{\pm 1})^4, (\zeta^{\pm 2})^4, (\zeta^{\pm 3})^2, (\zeta^{\pm 4})^4, (\zeta^{\pm 5})^2, (\zeta^{\pm 6})^2, (\zeta^{\pm 7})^2, (\zeta^{\pm 8})^4.$$

There exists an element y_1 in the algebraic group F_4 , cubing to x , with eigenvalues

$$1^8, (\theta^{\pm 1})^4, (\theta^{\pm 2})^4, (\theta^{\pm 3})^2, (\theta^{\pm 4})^4, (\theta^{\pm 5})^2, (\theta^{\pm 6})^2, (\theta^{\pm 7})^2, (\theta^{\pm 8})^2, (\theta^{\pm 9})^2$$

on V_{\min} , where θ is a primitive 51st root of unity with $\theta^3 = \zeta$. This preserves all eigenspaces except for the $\zeta^{\pm 8}$ ones, which lie in 2_4 and $4_{1,4}$. There also exists an element y_2 in F_4 , again cubing to x , and with eigenvalues

$$1^8, (\theta^{\pm 1})^4, (\theta^{\pm 2})^4, (\theta^{\pm 3})^2, (\theta^{\pm 4})^2, (\theta^{\pm 21})^2, (\theta^{\pm 22})^2, (\theta^{\pm 23})^2, (\theta^{\pm 24})^2, (\theta^{\pm 25})^4,$$

with now the $\zeta^{\pm 4}$ -eigenspaces being split and all others being preserved. Only the module 2_3 has ζ^4 as an eigenvalue, so the stabilizer of any simple submodule of $V_{\min} \downarrow_H$ contains either y_1 or y_2 , and hence is positive dimensional, either using the fact that $v(F_4) = 18$ or Proposition 4.10.

This proves that, regardless of the field k , or whether $N_{\bar{G}}(H)$ induces a field automorphism on H , $N_{\bar{G}}(H)$ is contained in a positive-dimensional subgroup stabilizing a σ -stable subspace of V_{\min} , as needed.

For $a = 5$, we use the traces of semisimple elements and get a unique set of composition factors,

$$32, (2_1, 2_2, 2_3, 2_4, 2_5)^2, 1^4.$$

An element of order 31 in H has 1-eigenspace of dimension 6 on these factors (since it has a 1-eigenspace of dimension 2 on the 32), hence is a blueprint for V_{\min} by Proposition 4.10. Thus H is a blueprint for V_{\min} , as claimed.

When $a = 6$, the group of field automorphisms has order 6, so we need to consider those factors that are stable under field automorphisms of orders 2 and 3. We start with those definable over \mathbb{F}_4 , i.e., stable under the automorphism of order 3. There are sixteen conspicuous sets of composition factors for elements of order up to 63, six of which have no factors of dimension 1 or 2, hence dealt with in Proposition 10.1, with the rest given by

$$8_{2,4,6}^4, (2_1, 2_3, 2_5)^2, 1^{12}, \quad (4_{1,3}, 4_{3,5}, 4_{1,5})^2, (2_1, 2_3, 2_5)^4, 1^8, \quad 8_{1,3,5}, (2_1, 2_3, 2_5)^8,$$

$$8_{1,3,5}, (4_{1,3}, 4_{3,5}, 4_{1,5})^2, 4_{1,4}, 4_{2,5}, 4_{3,6}, (2_1, 2_3, 2_5)^2, \quad 8_{1,3,5}, (4_{1,3}, 4_{3,5}, 4_{1,5})^2, 4_{1,6}, 4_{2,3}, 4_{4,5}, (2_1, 2_3, 2_5)^2,$$

up to field automorphism of H . The first two of these have more than six trivial composition factors, hence are blueprints for V_{\min} by Proposition 4.10; for the other three it can be checked manually that the elements of order 63 are blueprints for V_{\min} , by finding elements of order $315 = 5 \cdot 63$ that power to them and have the same number of eigenvalues on V_{\min} .

We also have to check sets of composition factors defined over \mathbb{F}_8 , where up to field automorphism there are ten conspicuous sets of composition factors for semisimple elements of order 21, three of which lose their conspicuousness on elements of order 63. The remaining seven all have at least eight trivial composition factors, hence are blueprints for V_{\min} by Proposition 4.10. These are

$$4_{1,4}^2, (2_1, 2_4)^8, 1^{16}, \quad 4_{3,6}^8, (2_1, 2_4)^2, 1^{16}, \quad (4_{1,5}, 4_{2,4})^2, 4_{1,4}^2, (2_1, 2_4)^4, (2_2, 2_5)^2, 1^8,$$

$$16_{1,3,4,6}^2, (2_1, 2_4)^2, (2_2, 2_5)^2, 1^8, \quad 16_{2,3,5,6}^2, (2_1, 2_4)^2, (2_3, 2_6)^2, 1^8,$$

$$(8_{2,3,6}, 8_{2,4,5})^2, 4_{2,5}^2, (2_1, 2_4)^2, 1^8, \quad (8_{1,3,6}, 8_{3,4,6})^2, 4_{1,4}^2, (2_1, 2_4)^2, 1^8$$

This completes the proof of the proposition. \square

We now can assume that k contains a splitting field for H and, moreover, that $N_{\bar{G}}(H) = HC_{\bar{G}}(H)$, as the composition factors of $V_{\min} \downarrow_H$, which is stable under $\text{Out}(G)$, are not compatible with an outer automorphism of H .

Proposition 10.3 Suppose that $V_{\min} \downarrow_H$ has at least one 2-dimensional composition factor and no trivial composition factors.

- (i) If $a = 3$ then H fixes a 2-space on V_{\min} .
- (ii) If $a = 4, 5, 6$ then H fixes a 2-space on V_{\min} or is a blueprint for V_{\min} .

Proof: Suppose that $a = 3$. Any 8s split off, so we just consider the 4s and 2s. The projective cover of $4_{i,i+1}$ is

$$P(4_{i,i+1}) = 4_{i,i+1}/2_{i+1}/1/2_{i-1}/1/2_{i+1}/4_{i,i+1},$$

and from this we see that no module can have a 2-dimensional composition factor, no trivial composition factor, and no have a 2-dimensional submodule or quotient. This proves (i).

For $a = 4$, we first compute the conspicuous sets of composition factors, finding eighty-one sets up to field automorphism of H . We have a list of those classes of elements of order 17 that are blueprints for V_{\min} , and all but fifteen of these sets appear on that list. We can also compute which have positive 2_i -pressure (or no 2_i) for every i , and find that only eighteen of these sets do. The intersection of these two short lists has just two sets of composition factors on it, and so we consider these two:

$$8_{1,2,3}, 4_{1,2}^2, 4_{1,3}^3, 4_{2,3}^3, 4_{2,4}, 2_1^2, 2_2^2, 2_3^2, \quad 8_{1,2,4}, 8_{2,3,4}, 4_{1,2}^2, 4_{1,3}, 4_{1,4}^2, 4_{2,3}^3, 2_1^2, 2_3^2.$$

The first of these must stabilize a 2-space on V_{\min} , as notice that otherwise the socle can consist of summands of $V_{\min} \downarrow_H$ and a submodule of $4_{1,2} \oplus 4_{1,3} \oplus 4_{2,3}$, but the largest submodules of those projectives with composition factors those in $V_{\min} \downarrow_H$ are

$$4_{2,4}/2_2/4_{1,2}, \quad 4_{1,3}/2_3/4_{2,3}/2_1, 2_3/4_{1,3}, \quad 2_3/4_{1,3}, 4_{2,3}/2_3, 8_{1,2,3}/4_{2,3},$$

so at most a single 2_1 can lie in $V_{\min} \downarrow_H$, a contradiction.

The second case is even easier, given that the corresponding submodules are

$$4_{1,2}/8_{1,2,4}/4_{1,2}, \quad 4_{1,3}/2_1/4_{1,4}, \quad 4_{1,3}/2_3/4_{2,3}.$$

This completes the proof for $a = 4$.

We now let $a = 5$. There are thirty possible multisets of dimensions for the composition factors of $V_{\min} \downarrow_H$ that have at least one 2, no 1s, and have the right trace of an element of order 3. If H does not fix a 2-space, then we need two 4s in the dimensions, removing twenty multisets of dimensions from the list. We can also apply Lemma 5.6, which shows that if there are no 8s in $V_{\min} \downarrow_H$ then we need at least as many 4s as 2s, removing another three. Since any 4 has 2-pressure at most 2, there needs to be more than half as many 4s as 2s in all cases; this brings us down to ten. These are

$$4^{10}, 2^8, \quad 8, 4^9, 2^6, \quad 16, 4^7, 2^6, \quad 8^2, 4^8, 2^4, \quad 16, 8, 4^6, 2^4, \\ 16^2, 4^4, 2^4, \quad 8^3, 4^7, 2^2, \quad 16, 8^2, 4^5, 2^2, \quad 32, 4^5, 2^2, \quad 16^2, 8, 4^3, 2^2.$$

In these cases we switch to proving that H is a blueprint for V_{\min} . (This could be done for the other cases but the amount of extra work is significant and so this has not been done.)

We give a table listing the total number of possible sets of composition factors for $V_{\min} \downarrow_H$, then those that are conspicuous, and finally those for which an element of order 31 in H is a blueprint for V_{\min} . These numbers are all up to a field automorphism of V_{\min} .

Case	Number	Conspicuous	31 is blueprint
$4^{10}, 2^8$	9145422	23	23
$8, 4^9, 2^6$	20420400	32	32
$16, 4^7, 2^6$	2402400	3	2
$8^2, 4^8, 2^4$	18718700	52	51
$16, 8, 4^6, 2^4$	3503500	12	8
$16^2, 4^4, 2^4$	150150	2	2
$8^3, 4^7, 2^2$	7550400	22	21
$16, 8^2, 4^5, 2^2$	1651650	20	19
$32, 4^5, 2^2$	6006	0	0
$16^2, 8, 4^3, 2^2$	99000	4	4

We now focus on these remaining eight conspicuous sets of composition factors:

$$\begin{aligned}
&16_{1,2,3,4}, 4_{2,5}, 4_{1,4}, 4_{2,4}^2, 4_{1,3}, 4_{1,5}, 4_{2,3}, 2_3^2, 2_2^2, 2_1^2, \quad 8_{2,3,5}, 8_{1,2,5}, 4_{1,4}, 4_{3,5}^2, 4_{1,3}, 4_{1,5}^2, 4_{2,3}^2, 2_3^2, 2_1^2, \\
&16_{1,3,4,5}, 8_{2,4,5}, 4_{2,5}, 4_{3,5}, 4_{1,5}, 4_{4,5}^2, 4_{2,3}, 2_3, 2_1^3, \quad 16_{1,3,4,5}, 8_{2,4,5}, 4_{1,4}, 4_{3,5}, 4_{1,3}, 4_{4,5}^2, 4_{2,3}, 2_3^2, 2_1^2, \\
&16_{1,2,3,4}, 8_{1,4,5}, 4_{2,5}, 4_{3,5}, 4_{2,4}, 4_{1,5}, 4_{3,4}, 4_{2,3}, 2_3^2, 2_1^2, \quad 16_{1,2,3,5}, 8_{2,3,4}, 4_{1,4}, 4_{3,5}, 4_{1,3}^2, 4_{4,5}, 4_{2,3}, 2_4, 2_3, 2_1^2, \\
&8_{2,4,5}, 8_{1,3,4}, 8_{1,2,5}, 4_{2,4}, 4_{1,5}, 4_{4,5}^2, 4_{3,4}, 4_{2,3}, 4_{1,2}, 2_3, 2_1, \quad 16_{1,2,4,5}, 8_{1,2,4}, 8_{1,2,5}, 4_{2,4}, 4_{1,5}, 4_{3,4}, 4_{2,3}, 4_{1,2}, 2_2, 2_1.
\end{aligned}$$

Recall from Lemma 5.1 that 2_i has extensions only with $4_{i,j}$ for $j \neq i, i+1$: from this we see that the third, fourth, fifth, seventh and eighth have non-positive 2_1 -pressure so have 2_1 s as submodules; the first case has 2_3 -pressure 0 so fixes a 2_3 submodule; the sixth case has 2_4 -pressure 0 so fixes a 2_4 submodule. This leaves the second case, which will fix a 2_1 submodule: we quotient out by any simple submodule other than $4_{1,5}$, $4_{1,4}$ and $4_{1,3}$, and take any submodule whose quotient is simple and not one of these, to leave a module W containing both 2_1 s and having only $4_{1,5}$ in the socle, except possibly for $4_{1,4}$ or $4_{1,3}$ appearing as a summand. We therefore consider the largest submodule of $P(4_{1,5})$ with composition factors from the factors of $V_{\min} \downarrow_H$, and this is

$$4_{1,4}/2_1/4_{1,5}.$$

Thus $V_{\min} \downarrow_H$ must have 2_1 as a submodule, and we are done for the case $a = 5$.

Finally, let $a = 6$. We have the same ten multisets of dimensions of composition factors for $V_{\min} \downarrow_H$ as the case of $a = 5$, and we perform the same analysis as before.

Case	Number	Conspicuous up to 21	Conspicuous up to 63	63 is blueprint
$4^{10}, 2^8$	420696342	68	41	41
$8, 4^9, 2^6$	1258472670	369	76	76
$16, 4^7, 2^6$	134306100	121	9	9
$8^2, 4^8, 2^4$	1410195600	1068	104	104
$16, 8, 4^6, 2^4$	244188000	750	38	38
$16^2, 4^4, 2^4$	7712064	89	12	12
$8^3, 4^7, 2^2$	626749200	983	90	90
$16, 8^2, 4^5, 2^2$	128200860	1097	80	80
$32, 4^5, 2^2$	244188	15	1	1
$16^2, 8, 4^3, 2^2$	5712000	208	24	24

As every conspicuous set of composition factors for $V_{\min} \downarrow_H$ has an element of order 63 that is a blueprint for V_{\min} , H is always a blueprint for V_{\min} , as needed. \square

Proposition 10.4 Suppose that $V_{\min} \downarrow_H$ has either two or four trivial composition factors.

- (i) If $a = 3$ then $V_{\min} \downarrow_H$ has a submodule of dimension at most 2.
- (ii) If $a = 4$ then either H fixes a line on $L(G)'$ or $N_{\bar{G}}(H)$ fixes a σ -stable, proper, non-zero subspace of V_{\min} whose stabilizer is positive dimensional.
- (iii) If $a = 5$ then $N_{\bar{G}}(H)$ fixes a σ -stable, proper, non-zero subspace of V_{\min} whose stabilizer is positive dimensional.
- (iv) If $a = 6$ then $N_{\bar{G}}(H)$ is either a blueprint for V_{\min} or fixes a subspace of dimension at most 2 of V_{\min} .

Proof: Let $a = 3$. As we have seen before, the projective cover of $4_{i,i+1}$ is

$$P(4_{i,i+1}) = 4_{i,i+1}/2_{i+1}/1/2_{i-1}/1/2_{i+1}/4_{i,i+1},$$

whence if $V_{\min} \downarrow_H$ has no 1- or 2-dimensional submodules or quotients, it is a sum of 8s and $P(4_{i,i+1})$ s for various i . In particular, since $\dim(P(4_{i,i+1})) = 16$, we have one of $P(4), 8^5$ and $P(4)^2, 8^3$, as there are between two and four trivial factors. Thus the composition factors of $V_{\min} \downarrow_H$ are either $8^5, 4^2, 2^3, 1^2$ or $8^3, 4^4, 2^6, 1^4$, on which an element of order 3 acts with trace -4 and -1 respectively, not a trace of an element of order 3 on V_{\min} , which is one of $-25, -7, 2, 20$. This completes the proof for $a = 3$.

Let $a = 4$. Using all semisimple elements, there are (up to field automorphism) 113 conspicuous sets of composition factors for $V_{\min} \downarrow_H$ with exactly two trivial composition factors. Only seventy-nine of these have corresponding factors on $L(G)$, and of these only thirty-eight have either no 2_i or positive 2_i -pressure for every i . One can eliminate three more as an element of order 17 is a blueprint for V_{\min} , leaving us with thirty-five. Two more of these have no 4-dimensional factors appearing with multiplicity greater than 1, so must stabilize either a line or 2-space as V_{\min} is self-dual.

We are left with thirty-three conspicuous sets of composition factors for $V_{\min} \downarrow_H$, still too many to list. Let W be the subquotient obtained from $V_{\min} \downarrow_H$ by quotienting out by the $\{8, 16\}$ -radical and taking the $\{8, 16\}$ -residual, and remove any 4-dimensional simple summands. Since H can be assumed not to fix a line or 2-space on V_{\min} , the socle of W consists of 4-dimensional modules, and the factors of $\text{soc}(W)$ consist of 4-dimensional simple modules that occur with multiplicity at least 2 in $V_{\min} \downarrow_H$, and hence W . Let

S_1, \dots, S_r be the 4-dimensional simple modules that appear in $V_{\min} \downarrow_H$ with multiplicity at least 2. (If a module appears more than twice, we take the floor of half of its multiplicity, since this is the maximum number of times it may appear in the socle.)

We construct the largest submodule W' of $P(S_1 \oplus \dots \oplus S_r)$ that consists solely of composition factors from $V_{\min} \downarrow_H$; certainly $W \leq W'$. Thus W' must have at least two trivial factors, and all the requisite 2-dimensional factors.

In fact, only twenty-two out of the thirty-three cases yield modules W' with any trivial factors, with five even being the zero module (as there are no such S_i). Another six can be removed for not having the correct 2-dimensional factors, leaving sixteen sets of composition factors. Five of these have corresponding set of composition factors on $L(G)'$ having pressure less than 6, hence H fixes a line on $L(G)'$ by Lemma 5.3 and the fact that an involution has at least six trivial Jordan blocks on $L(G)'$. The remaining eleven are as follows:

$$\begin{aligned} &8_{1,3,4}^2, 4_{1,2}, 4_{1,3}^2, 4_{1,4}^4, 2_1^4, 2_2, 1^2, & 8_{1,3,4}, 4_{1,3}^2, 4_{1,4}^3, 4_{2,3}^2, 4_{2,4}, 2_1^3, 2_2, 2_3^2, 2_4, 1^2, \\ &8_{1,3,4}, 4_{1,2}, 4_{1,3}, 4_{1,4}, 4_{2,3}, 4_{2,4}^2, 4_{3,4}^2, 2_1^3, 2_2, 2_3, 2_4^2, 1^2, & 8_{1,2,4}, 4_{1,2}^2, 4_{1,3}^2, 4_{1,4}, 4_{2,3}, 4_{2,4}, 4_{3,4}, 2_1^2, 2_2^2, 2_3^2, 2_4, 1^2, \\ &8_{2,3,4}, 4_{1,2}, 4_{1,3}^2, 4_{1,4}, 4_{2,3}^2, 4_{2,4}^2, 2_1^2, 2_2^2, 2_3^2, 2_4, 1^2, & 16, 4_{1,2}, 4_{1,3}, 4_{1,4}, 4_{2,4}^2, 4_{3,4}, 2_1^2, 2_2^2, 2_3^2, 2_4, 1^2, \\ &8_{1,2,3}, 8_{1,2,4}, 4_{1,2}^2, 4_{1,3}^2, 4_{1,4}^2, 4_{2,3}, 2_1^2, 2_2, 2_3^2, 1^2, & 8_{1,3,4}^2, 4_{1,2}, 4_{1,3}^2, 4_{1,4}^3, 4_{2,3}, 2_1^2, 2_2, 2_3^2, 1^2, \\ &8_{1,2,4}, 8_{1,3,4}, 4_{1,2}, 4_{1,3}, 4_{1,4}^2, 4_{2,3}^2, 4_{2,4}, 2_1^2, 2_2, 2_3, 2_4, 1^2, & 8_{1,2,4}^2, 8_{1,3,4}, 4_{1,3}^3, 4_{1,4}, 4_{2,3}, 2_1^2, 2_2, 2_3^2, 1^2, \\ &16, 8_{1,2,3}, 8_{1,3,4}, 4_{1,2}, 4_{1,3}, 4_{1,4}^2, 2_1^2, 2_2, 1^2. \end{aligned}$$

We can eliminate some more using module structures: in the second case, suppose that $4_{1,4}$ lies in the socle. If it is a summand, we quotient it out and ignore it, so suppose it is a submodule but not a summand, and quotient this out, also quotienting out any 2- and 1-dimensional factors that become submodules to produce a module U . This is a submodule of $P(4_{1,4})$ and so we use Lemma 5.7, seeing that U is a submodule of

$$2_1/1/2_2/1/2_1/4_{1,4};$$

if both trivials are in U then $V_{\min} \downarrow_H / U$ has 2_3 -pressure 0, so has 2_3 as a submodule, a contradiction from the definition of U . If there is a single trivial in U then firstly replace U by the 7-dimensional submodule $1/2_1/4_{1,4}$ of U , and since V_{\min} is self-dual, there is a (unique) corresponding submodule U' such that $V_{\min} \downarrow_H / U' \cong U^*$. If $U \leq U'$ then U'/U has no trivials and again it has 2_3 -pressure 0 and so we get a contradiction. Thus U' does not contain U , and we claim that in this case an involution u must act with exactly two trivial Jordan blocks, not allowed by [13, Table 7]. To see this, firstly let M denote the $\{1\}'$ -residual modulo the $\{1\}'$ -radical of $V_{\min} \downarrow_H$, so it is a submodule of $P(1)$, as otherwise it is simply $1^{\oplus 2}$, with this impossible by [13, Table 7]. The submodule U of $V_{\min} \downarrow_H$ has image inside M which is just $\text{soc}(M)$, and the image of U' has image inside M which is simply $\text{rad}(M)$. It is therefore clear that the image of U' contains the image of U and, since U is uniserial, U' contains U .

Thus we can remove any $4_{1,4}$ in the socle and top, perhaps remove two 2_1 s that are now in the socle and top, and assume that the resulting module V' is a self-dual submodule of $P(4_{1,3}) \oplus P(4_{2,3})$.

We now give the three modules obtained from the following procedure, given a socle S that is a submodule of $4_{1,3} \oplus 4_{2,3}$:

- (i) Take the preimage S_1 in $P(S)$ of the radical of $P(S)/S$ corresponding to all composition factors of $V_{\min} \downarrow_H$ other than those in S ;

- (ii) Take the preimage S_2 in $P(S)$ of the $\text{cf}(S)$ -radical of the quotient $P(S)/S_1$;
- (iii) Take the $\text{cf}(S)'$ -residual S_3 of S_2 .

This must contain the module V' , so we examine the composition factors of the modules S_3 for the choices of S , which are

$$4_{1,3}, 4_{1,3}/2_1, 2_3/4_{1,4}, 4_{2,3}/2_1, 2_3/4_{1,3}, \quad 4_{2,3}/2_3/1/2_4/1/2_3/4_{2,3},$$

$$4_{2,3}/2_3/1, 4_{1,3}/2_1, 2_4/1, 4_{1,3}, 4_{1,4}, 4_{2,3}/2_1, 2_3, 2_3/4_{1,3}, 4_{2,3}.$$

None of these has a 2_2 as a composition factor, and this yields a contradiction. Thus in the second case H must fix a line or 2-space on V_{\min} .

In the third case, W' might have enough 2-dimensional factors, but in order to have three 2_1 s in W' we need both $4_{2,4}$ and $4_{3,4}$ in the socle, whence they cannot appear elsewhere in the module (which they can do in our construction of W'). With this restriction, that $4_{2,4}$ and $4_{3,4}$ can only appear in the socle and top of $V_{\min} \downarrow_H$, we get the analogue of W' to be

$$4_{3,4}/2_3, 2_4/1, 1, 4_{1,3}, 4_{2,4}/2_1, 2_1, 2_2, 2_3/1, 1, 4_{1,2}, 4_{2,4}, 4_{3,4}/2_2, 2_4, 2_4/4_{2,4}, 4_{3,4},$$

which still does not have three 2_1 s in it, a contradiction.

In the fourth case, the socle of W cannot simply be $4_{1,2}$ as there are no 2_1 s in its contribution to W' . If it is $4_{1,2} \oplus 4_{1,3}$ then, arguing as in the previous case, we get

$$4_{1,2}/2_2/1, 4_{1,3}, 4_{1,3}/2_1, 2_3, 2_3/1, 4_{1,2}, 4_{1,4}, 4_{2,3}/2_1, 2_2, 2_3, 8_{1,2,4}/4_{1,2}, 4_{1,3},$$

and if it is just $4_{1,3}$ then we get

$$4_{1,3}, 4_{1,3}/2_1, 2_3/4_{1,4}, 4_{2,3}/2_1, 2_3/4_{1,3},$$

so in neither case can we contain all of W .

In the sixth case, the socle of W must be $4_{2,4}$, and if so no $4_{2,4}$ can appear outside of the socle and top of W . Taking the radical of $P(4_{2,4})/4_{2,4}$ with factors all other composition factors of $V_{\min} \downarrow_H$, then adding on as many $4_{2,4}$ s on top of that, then taking the $\{4_{2,4}\}'$ -residual of this (since the socle of W must be $4_{2,4}$), we end up with

$$4_{2,4}, 4_{2,4}/2_2, 2_4/4_{1,2}, 4_{3,4}/2_2, 2_4/4_{2,4},$$

clearly wrong. Thus in the sixth case H fixes a 1- or 2-space on V_{\min} .

In the ninth case, the possible factors of $\text{soc}(W)$ are $4_{1,4}$ and $4_{2,3}$, with both required for all of the 2-dimensional factors to be present, as an examination of W' proves. In this case, we do as above to find that W is a submodule of

$$4_{1,4}/2_1/1, 4_{1,4}/2_1, 2_2/1, 1, 4_{1,3}, 4_{1,4}/2_1, 2_3, 8_{1,3,4}/4_{1,4}, 4_{2,3},$$

which does not have a 2_4 , a contradiction.

For the other cases, the existence of the uniserial modules $M_1 = 4_{1,4}/2_1/1/2_2/1/2_1/4_{1,4}$ and $M_2 = 4_{1,3}/2_1/4_{1,4}/2_1/4_{1,3}$ prove directly that the first (M_1 plus M_2), fifth (M_1 twisted by the square of the field automorphism, M_2 untwisted, and M_2 twisted by the field), seventh (M_1 twisted by the field, plus M_2), eighth (M_1 plus M_2 twisted by the field squared), tenth and eleventh cases (both a single M_1) cannot be solved in the same way.

For these we will show that H fixes a (σ -stable) subspace whose stabilizer is positive dimensional, since it will contain an element of order $85 > v(E_7)$. In fact, five of the six satisfy an extra property that we will use later on: an element x of order 17 has seventeen distinct eigenvalues on V_{\min} , and has a preimage \hat{x} of order 85 that has eighteen eigenvalues on V_{\min} . This means that \hat{x} must have the same eigenspaces as x , except that the 1-eigenspace of x must split into two. Since only the trivial module has a 1-eigenspace for the action of x , this means that every submodule of $V_{\min} \downarrow_H$ that does not contain a trivial module must be stabilized by \hat{x} and hence by a positive-dimensional subgroup of G .

We are left with a single case to consider, which is

$$8_{1,3,4}^2, 4_{1,2}, 4_{1,3}^2, 4_{1,4}^3, 4_{2,3}, 2_1^2, 2_2, 2_3^2, 1^2;$$

if we choose x of order 17 and ζ a primitive 17th root of unity so that x acts on 2_1 with eigenvalues $\zeta^{\pm 1}$, then x acts on V_{\min} with eigenvalues

$$1^2, (\zeta^{\pm 1})^3, (\zeta^{\pm 2})^2, (\zeta^{\pm 3})^5, (\zeta^{\pm 4})^4, (\zeta^{\pm 5})^4, (\zeta^{\pm 6})^3, (\zeta^{\pm 7})^3, (\zeta^{\pm 8})^3,$$

and there is an element \hat{x} of order 85 in G that powers to x and has the same eigenspaces, except it splits the $\zeta^{\pm 1}$ and 1-eigenspaces, so has twenty eigenvalues on V_{\min} . An easy calculation shows that the only composition factors of V_{\min} on which x has 1 or $\zeta^{\pm 1}$ as an eigenvalue are 1, 2_1 and $4_{1,2}$: if 1 or 2_1 is a submodule of $V_{\min} \downarrow_H$ then its stabilizer is positive dimensional anyway, and if $4_{1,2}$ is a submodule then it is a summand, so there must be another factor in the socle, which therefore has a positive-dimensional stabilizer. This completes the proof of the proposition in the case where there are exactly two trivial factors in $V_{\min} \downarrow_H$.

Moving to exactly four trivial composition factors, up to field automorphism there are 114 conspicuous sets of composition factors for $V_{\min} \downarrow_H$. Twenty-two of these contain an element of order 17 that is a blueprint for V_{\min} , and twenty-five have no corresponding sets of composition factors for $L(G)$. Taken together, this leaves seventy sets of composition factors. Removing those with non-positive 1- or 2_i-pressure and those without two isomorphic 4-dimensional composition factors brings us down to fifty-one sets of factors.

As with the previous case, we construct the modules W and W' and apply the same test, reducing us to twenty-three sets of composition factors. Another three have pressure less than 6 on $L(G)'$, so fix a line on $L(G)'$ and can be discarded.

As we saw when considering two trivial factors, construction of the module W' does not take into account that if a 4-dimensional factor lies in the socle of W and has multiplicity exactly 2 in $V_{\min} \downarrow_H$ then it cannot appear anywhere other than the socle or the top of W . Including this, and ranging over all possible socles rather than just the largest one, yields a collection of modules for each case, all smaller than the original W' , and another twelve that no longer have enough 1- or 2-dimensional factors, bringing us down to eight. The last eight cases are as follows:

$$\begin{aligned} &4_{1,3}^3, 4_{1,4}, 4_{2,3}^3, 4_{2,4}, 2_1^4, 2_2^2, 2_3^3, 2_4, 1^4, & 4_{1,2}, 4_{1,3}^3, 4_{1,4}^2, 4_{2,3}, 4_{2,4}, 2_1^4, 2_2^2, 2_3^3, 2_4, 1^4, \\ &8_{1,3,4}, 4_{1,3}^2, 4_{1,4}^2, 4_{2,3}, 4_{3,4}^2, 2_1^3, 2_2, 2_3^2, 2_4^2, 1^4, & 8_{1,3,4}, 4_{1,2}, 4_{1,3}^2, 4_{1,4}^3, 4_{2,3}, 2_1^4, 2_2^2, 2_3^2, 1^4, \\ &8_{1,3,4}, 8_{2,3,4}, 4_{1,4}^3, 4_{2,4}, 4_{3,4}^2, 2_1^3, 2_2, 2_4^2, 1^4, & 8_{1,3,4}^2, 4_{1,4}^3, 4_{2,4}, 4_{3,4}^2, 2_1^3, 2_2, 2_4^2, 1^4, \\ &8_{1,2,3}, 8_{1,3,4}, 4_{1,2}, 4_{1,3}, 4_{1,4}^2, 4_{2,3}^2, 2_1^2, 2_2, 2_3^2, 2_4, 1^4, & 8_{1,3,4}^2, 4_{1,3}, 4_{1,4}^3, 4_{2,3}, 4_{3,4}, 2_1^3, 2_2^2, 2_3, 1^4. \end{aligned}$$

In the first case, the socle of W can be either $4_{1,3}$ or $4_{1,3} \oplus 4_{2,3}$. If the socle of W is $4_{1,3}$ then the module W' in which W can be found is

$$\begin{array}{c}
4_{1,3} \\
2_1 \ 2_3 \\
1 \ 4_{1,4} \ 4_{2,3} \\
2_1 \ 2_2 \ 2_3 \ 2_4 \\
1 \ 1 \ 4_{1,3} \ 4_{1,3} \ 4_{2,4} \\
2_1 \ 2_2 \ 2_3 \ 2_4 \\
1 \ 4_{1,4} \ 4_{2,3} \\
2_1 \ 2_3 \\
4_{1,3}
\end{array}$$

This is self-dual, so has a simple top, and since it is 64-dimensional, W must be contained in $\text{rad}(W')$, and indeed in the $\{4_{1,3}, 4_{2,3}\}'$ -residual of this, which is

$$4_{2,3}/2_3/1, 4_{1,3}, 4_{1,3}/2_1, 2_3, 2_4/1, 4_{1,4}, 4_{2,3}/2_1, 2_3/4_{1,3},$$

which has no 2_2 , so $4_{1,3}$ cannot be the socle. If $4_{1,3} \oplus 4_{2,3}$ is the socle, then the module W' is the sum of the one above and

$$4_{1,3}, 4_{2,3}/2_1, 2_3/1, 4_{1,4}/2_1, 2_4/1, 4_{1,3}/2_3/4_{2,3},$$

which also has no 2_2 . The same statement about the top $4_{1,3}$ not appearing in W remains true, and so we take the same residual (this is why we took the $\{4_{1,3}, 4_{2,3}\}'$ -residual rather than the $\{4_{1,3}\}'$ -residual above) and see no 2_2 again. Thus H must fix a 1- or 2-space on V_{\min} .

In the second case, the socle of W' must be $4_{1,3}$, and indeed W' is the same module as in the previous case, so the same method works there.

In the third case, the socle of W' must be $4_{1,3} \oplus 4_{1,4} \oplus 4_{3,4}$. We can construct such a module, namely

$$\begin{array}{ccccccc}
& & 4_{1,4} & & 4_{3,4} & & \\
4_{1,3} & & 2_1 & & 2_4 & & \\
2_3 & & 1 & & 1 & & \\
4_{2,3} \oplus & 2_2 \oplus & 2_1 \oplus & 8_{1,3,4} & , & & \\
2_3 & & 1 & & 1 & & \\
4_{1,3} & & 2_1 & & 2_4 & & \\
& & 4_{1,4} & & 4_{3,4} & &
\end{array}$$

so we will need to look at elements of G to solve this case.

In the fourth case, the module W' is the self-dual module

$$\begin{array}{c}
4_{1,4} \\
2_1 \ 8_{1,3,4} \\
1 \ 4_{1,3} \ 4_{1,4} \\
2_1 \ 2_2 \ 2_3 \\
1 \ 1 \ 4_{2,3} \\
2_1 \ 2_2 \ 2_3 \\
1 \ 4_{1,3} \ 4_{1,4} \\
2_1 \ 8_{1,3,4} \\
4_{1,4}
\end{array}$$

which has two $8_{1,3,4}$ s, so we can as in the first two cases take the $\{4_{1,4}\}'$ -residual of $\text{rad}(W')$ to get a module

$$4_{1,4}/2_1/1/2_2/1, 4_{1,4}/2_1, 8_{1,3,4}/4_{1,4},$$

which cannot work for several reasons, so that H fixes a line or 2-space on V_{\min} . The exact same module appears as W' in the eighth case as well, so this method works there.

In the fifth and sixth cases, W' must have socle $4_{1,4} \oplus 4_{3,4}$ and we can construct a module with the right composition factors, namely

$$\begin{array}{ccc} 4_{1,4} & & 4_{3,4} \\ 2_1 & & 2_4 \\ 1 & & 1 \\ 2_2 & \oplus & 2_1 \\ 1 & & 1 \\ 2_1 & & 2_4 \\ 4_{1,4} & & 4_{3,4} \end{array}$$

with the remaining factors being summands.

In the seventh case, W' must have $4_{1,4} \oplus 4_{2,3}$ in the socle, and a module with the right composition factors has the first summand above and the second summand twisted by three iterations of the field automorphism, so

$$\begin{array}{ccc} 4_{1,4} & & 4_{2,3} \\ 2_1 & & 2_3 \\ 1 & & 1 \\ 2_2 & \oplus & 2_4 \\ 1 & & 1 \\ 2_1 & & 2_3 \\ 4_{1,4} & & 4_{2,3} \end{array}$$

We have therefore eliminated the first, second, fourth and eighth cases, and will look at semisimple elements in the third, fifth, sixth and seventh cases.

In the third case, the element x acts on V_{\min} with eigenvalues

$$1^4, (\zeta^{\pm 1})^3, (\zeta^{\pm 2})^2, (\zeta^{\pm 3})^3, (\zeta^{\pm 4})^5, (\zeta^{\pm 5})^5, (\zeta^{\pm 6})^2, (\zeta^{\pm 7})^2, (\zeta^{\pm 8})^4,$$

and there exists an element of order 85 that powers to x and has nineteen distinct eigenvalues on V_{\min} , only splitting the $\zeta^{\pm 2}$ -eigenspaces. In $V_{\min} \downarrow_H$, these lie in the 2_2 and $4_{2,3}$, the latter of which can only lie in the socle if it is a summand. Therefore every other simple submodule is preserved by an element of order $85 > v(E_7)$ and therefore a positive-dimensional subgroup of G .

We do the same thing in the fifth case, finding an element of order 85 that powers to x and only disturbs the 1- and $\zeta^{\pm 1}$ -eigenspaces. Since these only lie in the trivial and 2_1 , every simple submodule of $V_{\min} \downarrow_H$ is stabilized by a positive-dimensional subgroup of G .

For the sixth case, there are eight elements of order 85 that power to x and have nineteen eigenvalues on V_{\min} : four split the $\zeta^{\pm 6}$ -eigenspace and the other four split the $\zeta^{\pm 8}$ -eigenspace. The $\zeta^{\pm 6}$ -eigenspace is contributed to by $4_{2,4}$ and $8_{1,3,4}$ from $V_{\min} \downarrow_H$, and the $\zeta^{\pm 8}$ -eigenspace is contributed to by 2_4 and $4_{1,4}$. Thus any simple submodule of $V_{\min} \downarrow_H$ is stabilized by at least one element of order 85, and so the stabilizer is positive dimensional, as claimed.

Finally, we have the seventh case. Here, the smallest number of eigenvalues that an element of order 85 powering to x has on V_{\min} is twenty-three, but there is one that splits the $\zeta^{\pm 1}$, $\zeta^{\pm 2}$ and $\zeta^{\pm 5}$ -eigenspaces. The

element x has eigenvalues $\zeta^{\pm 7}, \zeta^{\pm 8}$ on $4_{1,4}$, which is in the socle of $V_{\min} \downarrow_H$ as we saw above. (We actually saw that it was in the socle of W , but all modules that appear with multiplicity greater than 1 are either in the socle of W or have dimension at most 2, so that $\text{soc}(W) \leq \text{soc}(V_{\min} \downarrow_H)$ or H fixes a 1- or 2-space on V_{\min} .) Thus we see that an element of order 85, and hence a positive-dimensional subgroup of G , stabilizes the $(\sigma$ -stable) submodule $4_{1,4}$ of $V_{\min} \downarrow_H$, completing the case for $a = 4$.

Let $a = 5$, and firstly suppose that there are two trivial composition factors, so we need at least three 2s and two 4s, to avoid fixing a line or 2-space on V_{\min} . There are seventeen possible sets of dimensions of composition factors with these properties that also have the correct trace of an element of order 3: if there are exactly two 4s in $V_{\min} \downarrow_H$ then we can use Lemma 5.7 to see that we can have exactly three 2s, thus eliminating two of these cases, and if there are three 4s we can have at most eight 2s, eliminating two more.

We now give a table listing the possible sets of dimensions, together with the number of sets of composition factors (up to field automorphism) with those dimensions, those that are conspicuous, those for which the element x of order 31 is a blueprint for V_{\min} , and those for which there exists an element \hat{x} of order 93, cubing to x , and such that \hat{x} has one more distinct eigenvalue than x . This last condition does not ensure that H is a blueprint for V_{\min} , but does show that H lies inside a positive-dimensional subgroup stabilizing every simple submodule of $\text{soc}(V_{\min} \downarrow_H)$ not of dimension 1 or 32.

To see this, if \hat{x} has one more eigenvalue than x then, since \hat{x} must be real as it lies in E_7 , all eigenspaces are preserved except for the 1-eigenspace. As only the trivial and 32-dimensional have 1 as an eigenvalue for x , this means that \hat{x} , and a positive-dimensional subgroup of G stabilizing the same subspaces as \hat{x} , fix any simple submodule of $V_{\min} \downarrow_H$ not of dimension 1 or 32. In particular, this proves that H lies inside a member of \mathcal{X}^σ .

Case	Number	Conspicuous	31 is blueprint	One more eigenvalue
$4^6, 2^{15}, 1^2$	3879876	5	4	0
$8, 4^5, 2^{13}, 1^2$	9529520	2	2	0
$8^2, 4^4, 2^{11}, 1^2$	10735725	13	12	1
$4^9, 2^9, 1^2$	6952660	16	12	0
$8, 4^8, 2^7, 1^2$	16044600	30	23	0
$16, 4^6, 2^7, 1^2$	1651650	9	3	1
$8^2, 4^7, 2^5, 1^2$	15855840	54	29	12
$16, 8, 4^5, 2^5, 1^2$	2522520	24	10	3
$16^2, 4^3, 2^5, 1^2$	83160	6	6	0
$8^3, 4^6, 2^3, 1^2$	7707700	22	5	9
$16, 8^2, 4^4, 2^3, 1^2$	1376375	19	14	3
$32, 4^4, 2^3, 1^2$	5005	1	0	0
$16^2, 8, 4^2, 2^3, 1^2$	57750	3	1	1

Excluding both those that are blueprints and where there is an element with one more eigenvalue, we are left with forty-eight conspicuous sets of composition factors. Twenty of these forty-eight have no corresponding set of composition factors on $L(G)'$, so cannot yield embeddings of H into G .

We are left with twenty-eight conspicuous sets of composition factors for $V_{\min} \downarrow_H$, still too many to list. Let W be the subquotient obtained from $V_{\min} \downarrow_H$ by quotienting out by the $\{8, 16, 32\}$ -radical and taking the $\{8, 16, 32\}$ -residual, and remove any 4-dimensional simple summands. Since H can be assumed not to fix a line or 2-space on V_{\min} , the socle of W consists of 4-dimensional modules, and the factors of $\text{soc}(W)$

consist of 4-dimensional simple modules that occur with multiplicity at least 2 in $V_{\min} \downarrow_H$, and hence W . Let S_1, \dots, S_r be the 4-dimensional simple modules that appear in $V_{\min} \downarrow_H$ with multiplicity at least 2. (Note that no composition factor of $V_{\min} \downarrow_H$, in the twenty-eight remaining sets of factors, appears with multiplicity greater than 3, so we need only one copy of each S_i .)

We construct the largest submodule W' of $P(S_1 \oplus \dots \oplus S_r)$ that consists solely of composition factors from $V_{\min} \downarrow_H$; certainly $W \leq W'$. Thus W' must have at least two trivial factors, and all the requisite 2-dimensional factors. In fact, only nine out of the twenty-eight cases yield modules W' with any trivial factors, with ten even being the zero module (as there are no such S_i). Another seven can be removed for not having the correct 2-dimensional factors, leaving the following two sets of factors:

$$8_{1,3,5}, 8_{1,4,5}, 4_{1,4}, 4_{1,3}^2, 4_{1,5}^2, 4_{2,3}, 4_{1,2}, 2_3^2, 2_2, 2_1^2, 1^2, \quad 16_{1,3,4,5}, 8_{1,3,4}, 8_{1,4,5}, 4_{1,4}, 4_{1,5}^2, 4_{1,2}, 2_2, 2_1^2, 1^2.$$

In these final two cases we need to consider a preimage \hat{x} that does not stabilize all eigenspaces, but does stabilize those that make up some submodule of $V_{\min} \downarrow_H$. In the first case, x has thirty-one eigenvalues on V_{\min} , and the fewest number of eigenvalues for a preimage \hat{x} of order 93 is thirty-five (two such preimages, each a power of the other), with the four eigenvalues of x not being stabilized being $\zeta^{\pm 14}, \zeta^{\pm 15}$ where ζ is a primitive 31st root of unity.

In the second case, x again has thirty-one eigenvalues and the fewest number of eigenvalues for \hat{x} is thirty-four (four preimages, yielding two subgroups of order 93), with the three eigenvalues of x not stabilized being either $1, \zeta^{\pm 2}$ or $1, \zeta^{\pm 3}$, depending on the choice of preimage.

The eigenvalues of x on $4_{1,2}$ are $\zeta^{\pm 1}, \zeta^{\pm 3}$, so if this is a submodule (hence summand) of $V_{\min} \downarrow_H$ then that submodule is stabilized by a positive-dimensional subgroup of G , as needed. There are no extensions between $4_{1,2}$ and any of $4_{1,3}, 4_{1,5}, 2_1$ or 2_3 , and so since all other composition factors are multiplicity free, and $4_{1,2}$ is multiplicity free, there can be no extensions between $4_{1,2}$ and any other composition factor, as V_{\min} is self-dual. Thus $4_{1,2}$ splits off in both cases, and our result is proved for two trivial factors.

If there are four trivial composition factors in $V_{\min} \downarrow_H$, then there are twenty-eight possible sets of dimensions for the factors of $V_{\min} \downarrow_H$ that have a good trace of an element of order 3, and we exclude those that do not have at least five 2s – bringing us down to sixteen sets – and those that do not have three 4s as needed by Proposition 5.7. We apply this lemma again to see that we cannot have too many 2s per 4, and this brings us down to six possible sets of dimensions, given in the table below.

Case	Number	Conspicuous	31 is blueprint
$4^5, 2^{16}, 1^4$	1939938	3	2
$4^8, 2^{10}, 1^4$	4866862	14	14
$8, 4^7, 2^8, 1^4$	11325600	30	28
$16, 4^5, 2^8, 1^4$	990990	7	3
$8^2, 4^6, 2^6, 1^4$	11561550	45	45
$16, 8, 4^4, 2^6, 1^4$	1501500	19	18

This leaves just six sets of composition factors that are not guaranteed to be blueprints for $V_{\min} \downarrow_H$. These are

$$\begin{aligned} &8_{1,4,5}, 4_{3,5}^2, 4_{1,3}^2, 4_{1,5}^2, 4_{2,3}, 2_4^2, 2_3^2, 2_2^2, 2_1^2, 1^4, \quad 16_{1,2,3,5}, 4_{2,5}^2, 4_{1,3}^2, 4_{2,3}, 2_4^2, 2_3^2, 2_2^2, 2_1^2, 1^4, \\ &16_{1,2,3,4}, 4_{1,4}, 4_{1,3}, 4_{1,5}, 4_{3,4}^2, 2_5, 2_4^2, 2_3, 2_2^2, 2_1^2, 1^4, \quad 16_{1,3,4,5}, 4_{1,4}, 4_{1,5}, 4_{4,5}^2, 4_{2,3}, 2_5^2, 2_3^2, 2_2, 2_1^3, 1^4, \\ &16_{1,2,4,5}, 4_{2,5}, 4_{3,5}, 4_{2,4}^2, 4_{3,4}, 2_5^2, 2_4^2, 2_2, 2_1^3, 1^4, \quad 16_{1,2,3,4}, 8_{1,3,4}, 4_{1,4}, 4_{2,4}, 4_{3,4}, 4_{2,3}, 2_5, 2_3, 2_2^2, 2_1^2, 1^4. \end{aligned}$$

They all have corresponding sets of composition factors on $L(G)$, but the easiest way to eliminate them is to consider the modules W and W' from the case of two trivial factors: in each of the six cases, we have at most two trivial factors in W' , and so H must always fix a line or 2-space on V_{\min} , as needed.

When $a = 6$, we have exactly the same possible dimensions for composition factors for $V_{\min} \downarrow_H$ as for $a = 5$. The traces of semisimple elements of order up to 21 are known, but not 63 or 65, so we can check if a set of composition factors are conspicuous up to 21. Letting x be an element of order 63 in H , we note that we have a list of all semisimple elements of order 21 in G , but not 63, so we use the preimage trick from Section 4.2 firstly to see if the composition factors are conspicuous up to 63, and then use the preimage trick again to see if there exists an element \hat{x} of order $63 \cdot 5 = 195$ with the same eigenspaces as x and with $\hat{x}^5 = x$. In every case, we find that the element of order 63 is a blueprint for V_{\min} .

Case	Number	Conspicuous up to 21	Conspicuous up to 63	63 is blueprint
$4^6, 2^{15}, 1^2$	100155870	6	6	6
$8, 4^5, 2^{13}, 1^2$	332095680	22	3	3
$8^2, 4^4, 2^{11}, 1^2$	467812800	60	18	18
$4^9, 2^9, 1^2$	272669110	164	21	21
$8, 4^8, 2^7, 1^2$	844192800	1201	40	40
$16, 4^6, 2^7, 1^2$	76744800	254	16	16
$8^2, 4^7, 2^5, 1^2$	1025589600	3079	93	93
$16, 8, 4^5, 2^5, 1^2$	146512800	1203	59	59
$16^2, 4^3, 2^5, 1^2$	3427200	53	20	20
$8^3, 4^6, 2^3, 1^2$	557110500	2665	54	54
$16, 8^2, 4^4, 2^3, 1^2$	89964000	996	63	63
$32, 4^4, 2^3, 1^2$	171360	14	5	5
$16^2, 8, 4^2, 2^3, 1^2$	2688000	58	18	18

Case	Number	Conspicuous up to 21	Conspicuous up to 63	63 is blueprint
$4^5, 2^{16}, 1^4$	39437442	4	4	4
$4^8, 2^{10}, 1^4$	160048350	170	19	19
$8, 4^7, 2^8, 1^4$	498841200	792	47	47
$16, 4^5, 2^8, 1^4$	37414170	61	12	12
$8^2, 4^6, 2^6, 1^4$	626754246	1484	85	85
$16, 8, 4^4, 2^6, 1^4$	70686000	146	37	37

This completes the proof for $3 \leq a \leq 6$, as needed. \square

We are left with H having at least six trivial composition factors, where by the remarks at the start of this subsection we noted that if $a = 5, 6$ then H is always a blueprint for V_{\min} .

Proposition 10.5 Suppose that $a \geq 3$ and $V_{\min} \downarrow_H$ has at least six trivial composition factors.

- (i) If $a = 3$ then $V_{\min} \downarrow_H$ has a 1- or 2-dimensional submodule or $V_{\min} \downarrow_H$ is

$$8 \oplus P(4_{1,2}) \oplus P(4_{2,3}) \oplus P(4_{1,3}).$$

- (ii) If $a = 4$ then H is a blueprint for V_{\min} or H fixes a subspace of dimension at most 2 on V_{\min} .

(iii) If $a \geq 5$ then H is a blueprint for V_{\min} .

Proof: (iii) follows from the remarks above. For (i) we use the proof of the previous proposition to note that the only possibility is that $V_{\min} \downarrow_H$ is the sum of three $P(4)$ s and an 8, so we consider the ten possible such modules, and note that only one has a conspicuous set of composition factors for $V_{\min} \downarrow_H$, the one mentioned.

We are left with $a = 4$. Here we use only non-blueprint elements of order 17 to restrict the number of possibilities. We also assume that $V_{\min} \downarrow_H$ has positive pressure, else H fixes a line on V_{\min} , and has at least two 4s, else it would fix a 2-space on V_{\min} .

Remove any 8s and 16s in the top and socle of $V_{\min} \downarrow_H$, together with any simple summands of dimension 4, leaving a self-dual module W whose top and socle consist of 4-dimensional modules, with W having all trivial factors in $V_{\min} \downarrow_H$.

The projectives $P(4_{1,2})$ and $P(4_{1,3})$ both have exactly four trivial composition factors, and have dimension 64. Therefore we cannot have the whole projective, so remove the simple top, then any 1-, 2- and 8-dimensional modules from the top to find the following modules:

$$4_{1,2}, 4_{2,4}/2_2, 2_4/1, 4_{3,4}/2_3, 2_4/1, 4_{1,2}, 4_{2,4}/2_2, 8_{1,2,4}/4_{1,2};$$

$$4_{1,4}, 4_{2,3}/2_1, 2_3/1, 1, 4_{1,3}, 4_{1,3}, 4_{2,4}/2_1, 2_2, 2_3, 2_4/1, 4_{1,4}, 4_{2,3}/2_1, 2_3/4_{1,3}.$$

From this we see that we need at least two 4s in the socle, and can have two only if they are both $4_{1,3}$ or $4_{2,4}$. This means that we need either three different 4s appearing in $V_{\min} \downarrow_H$ with multiplicity at least two, or $4_{1,3}^4, 4_{2,4}^4$ or $4_{1,3}^2, 4_{2,4}^2$.

Using the traces of non-blueprint semisimple elements of order 17, and traces of all elements of order 3, 5 and 15, we end up with, up to field automorphism, ten conspicuous sets of composition factors with at least six trivials, positive pressure, and at least two 4s. These are

$$4_{1,3}^2, 4_{1,4}^2, 4_{2,3}^2, 2_1^4, 2_2^2, 2_3^4, 2_4^2, 1^8, \quad 4_{1,3}^2, 4_{1,4}^2, 4_{2,3}, 4_{3,4}^2, 2_1^4, 2_2^3, 2_3^2, 2_4^2, 1^6, \quad 8_{2,3,4}, 4_{1,3}^2, 4_{1,4}^2, 4_{3,4}^2, 2_1^4, 2_2^3, 2_4^2, 1^6,$$

$$4_{1,3}^3, 4_{1,4}, 4_{2,3}^3, 2_1^4, 2_2^2, 2_3^4, 2_4, 1^6, \quad 8_{1,3,4}, 4_{1,3}^3, 4_{1,4}^2, 4_{2,4}, 2_1^4, 2_2, 2_3^2, 2_4^2, 1^6, \quad 8_{1,2,4}, 8_{2,3,4}, 4_{1,3}^3, 4_{2,4}^2, 2_1^4, 2_2, 2_3^2, 1^6,$$

$$16, 4_{1,3}, 4_{1,4}, 4_{3,4}^2, 2_1^3, 2_2^3, 2_3, 2_4^2, 1^6, \quad 8_{2,3,4}, 4_{1,2}, 4_{1,3}, 4_{2,3}^2, 4_{3,4}^2, 2_1^3, 2_2^2, 2_3^2, 2_4^2, 1^6,$$

$$8_{1,3,4}^2, 4_{1,3}^2, 4_{1,4}, 4_{2,3}, 4_{2,4}, 2_1^3, 2_3^2, 2_4^2, 1^6, \quad 8_{1,2,3}^2, 4_{1,2}, 4_{1,3}, 4_{1,4}, 4_{2,3}^2, 2_1^2, 2_2^2, 2_3^2, 2_4, 1^6.$$

By our previous remarks, in all but the first three cases H must fix either a 1-space or a 2-space on V_{\min} , as needed. In those three cases, all 4s that appear with multiplicity greater than 1 must appear in the socle of $V_{\min} \downarrow_H$.

The first case we saw before in Proposition 10.2, but we will come back to it. In the second case we take the preimages of the $\{1, 2, 4_{2,3}\}$ -radicals of $P(4_{1,3})/4_{1,3}$, $P(4_{1,4})/4_{1,4}$ and $P(4_{3,4})/4_{3,4}$ to produce three modules in whose direct sum $\text{rad}(W)$ is a submodule, but these are

$$1/2_2, 2_3, 2_4/1, 4_{2,3}/2_1, 2_3/4_{1,3}, \quad 2_1/1/2_2/1/2_1/4_{1,4}, \quad 2_4/1/2_1/1/2_4/4_{3,4},$$

and $\text{rad}(W)$ cannot have a trivial quotient, so there are only five 1s in W , a contradiction. In the third case we do the same thing, but with the $\{1, 2_1, 2_2, 2_4, 8_{2,3,4}\}$ -radicals, to get

$$2_1/4_{1,3}, \quad 2_1/1/2_2/1/2_1/4_{1,4}, \quad 2_4/1/2_1/1/2_4, 8_{2,3,4}/4_{3,4},$$

and clearly we have a contradiction here. Back to the first case, the appropriate modules here are

$$2_2, 2_4/1/2_1, 2_3/4_{1,3}, \quad 2_1/1/2_2/1/2_1/4_{1,4}, \quad 2_3/1/2_4/1/2_3/4_{2,3},$$

so we again must stabilize a 2-space on V_{\min} .

□

11 E_7 in odd characteristic: PSL_2 embedding

In this section, k is a field of characteristic $p \geq 3$ and $G = E_7(k)$, by which we mean the simply connected form, i.e., $|Z(G)| = 2$ and $G' = G$. Let \bar{G} be an almost simple group with socle $G/Z(G)$. From [10] we see that $v(E_7) = 75$ for odd integers, so if H is any subgroup of G with a semisimple element of odd order 77 or more, then H is a blueprint for V_{\min} . In addition, in [9] we prove that $\mathrm{PSL}_2(9)$ cannot be a maximal subgroup of \bar{G} either, so here we let $H = \mathrm{PSL}_2(p^a)$ with $a = 3, 4$ if $p = 3$ and $p^a \leq 150 = 2 \cdot v(E_7)$ if $p \geq 5$. Let $L = \mathrm{PSL}_2(p) \leq H$ and let u denote a unipotent element of L of order p .

By Proposition 4.10, if a semisimple element x has order at least 31 in G and centralizes a 6-space on V_{\min} , then x is a blueprint for V_{\min} . Since any semisimple element in H has a 1-dimensional 1-eigenspace on every odd-dimensional simple module, if H has at least six odd-dimensional composition factors on V_{\min} and then $p^a \geq 60$ then H is a blueprint for V_{\min} . This normally ends up being the case.

11.1 Characteristic 3

Now let $H = \mathrm{PSL}_2(3^a)$ for some $a \geq 1$. Since $v(E_7) = 75$, we assume that $a \leq 4$, and from [9] we assume that $a \neq 2$, so $a = 3, 4$. If $a = 4$ then we may assume that there are fewer than six odd-dimensional composition factors in $V_{\min} \downarrow_H$, by the discussion at the start of this section.

We begin by computing the composition factors of $V_{\min} \downarrow_L$, which depends only on the trace of an involution, ± 8 . This means that there are eight more of one factor than the other, so $3^{12}, 1^{20}$ and $3^{16}, 1^8$. From Lemma 5.8 we can see the possible dimensions of composition factors for $V_{\min} \downarrow_H$: if $V_{\min} \downarrow_L$ has factors $3^{16}, 1^8$ then we must have at least eight 3-dimensional factors in $V_{\min} \downarrow_H$, and if the factors are $3^{12}, 1^{20}$ then as only 9 and 1 for H have more 1s than 3s on restriction to L , we need at least eight of these in $V_{\min} \downarrow_H$, and again have at least eight odd-dimensional composition factors in $V_{\min} \downarrow_H$. This gives us the first proposition.

Proposition 11.1 Let $p = 3$ and $a = 4$. A semisimple element of order 41 in H is always a blueprint for V_{\min} , and hence H is always a blueprint for V_{\min} .

We turn to $a = 3$, where we cannot quite get the same result, but we come close.

Proposition 11.2 Let $p = 3$ and $a = 3$. Either H is a blueprint for V_{\min} or H fixes a line on either V_{\min} or $L(G)$.

Proof: As with F_4 and E_6 , we want to discount conspicuous sets of composition factors where a semisimple element is a blueprint for V_{\min} . We already know that there are 97 classes of semisimple elements of order 13 that are blueprints for the minimal module for F_4 , and there are 188 classes of semisimple elements of order 13 in E_7 whose 1-eigenspace is at least 8-dimensional, leaving 91 classes to which an element of order 13 in H can belong.

Using this, we find up to field automorphism eight conspicuous sets of composition factors, two of which have negative pressure so will not be displayed. The other six are

$$\begin{aligned} &9_{1,3}, 4_{1,3}^9, 3_1, 1^8, & 4_{1,2}^4, 4_{1,3}^4, 4_{2,3}^4, 1^8, & 9_{2,3}^3, 4_{1,2}^5, 4_{1,3}, 1^5, \\ &4_{1,2}^6, 3_1^9, 3_2, 1^2, & 4_{1,2}, 4_{1,3}^5, 4_{2,3}, 3_1^9, 1, & 9_{2,3}, 4_{1,2}^5, 3_1^5, 3_2^4. \end{aligned}$$

The first and third have pressure 1 so fix a line on V_{\min} by Lemma 5.13. The second case fixes a line on V_{\min} by Lemma 5.12.

For the fourth, if H does not fix a line on V_{\min} then we may assume that the socle consists of $4_{1,2}$ s by quotienting out any 3s in the socle, and the $\{1, 3_1, 3_2, 4_{1,2}\}$ -radical of $P(4_{1,2})$ is

$$4_{1,2}/1, 3_2/4_{1,2},$$

but since there is only one 3_2 in $V_{\min} \downarrow_H$ we cannot cover both trivials in this way, thus H fixes a line on V_{\min} . (Alternatively, the factors of H on $L(G)$ are $4_{1,2}^{16}, 3_1^{10}, 3_2^6, 1^{21}$, so H fixes a line on $L(G)$.)

The fifth case is $4_{1,2}, 4_{2,3}, 4_{1,3}^5, 3_1^9, 1$, which yields a set of composition factors on $L(G)$ of

$$4_{1,2}^5, 4_{2,3}^5, 4_{1,3}^{11}, 3_1^{10}, 3_3, 1^{16},$$

which has pressure 5, and so might not have a trivial submodule or quotient. However, the largest submodule of $P(4_{i,j})$ with these composition factors has three trivial composition factors in all cases, and so we need at least six 4s in the socle of $L(G) \downarrow_H$ (once we remove all 3s), contradicting the fact that the module has pressure 5, so we fix a line.

The final case is $9_{1,2}, 4_{1,2}^5, 3_1^5, 3_2^4$. The 9 splits off and we may quotient out by the $\{3_1, 3_2\}$ -radical to get a module with $4_{1,2}$ s in the socle. On this we can only place 3_2 s, and so u would act on V_{\min} as $3^{17}, 1^5$, not a valid unipotent action in [13, Table 7], so H cannot embed with these factors. \square

11.2 Characteristic at least 5

We now let $p \geq 5$, let $H = \text{PSL}_2(p^a)$ with $a \geq 1$, let $L = \text{PSL}_2(p) \leq H$ and let $u \in L$ have order p . We begin by producing a list of all unipotent classes to which u can belong, excluding those that come from generic classes (see Lemma 4.6) and those that fail Lemma 5.15. Moreover, we make a few remarks now about indecomposable modules for L , which can cut down our list.

When $p = 5, 13, 17$, we use Corollary 5.18, so for these primes the number of blocks of each even size less than $p - 1$ is even. For $p = 7, 11, 19, 23$, there exists a unique indecomposable module for L of dimension congruent to a given even number modulo p .

For $p = 11$, the self-dual module of dimension congruent to 6 modulo p has socle structure

$$1, 3, 5, 7, 9/1, 3, 5, 7, 9$$

and has dimension 50. The trace of an involution on the module is 0, and since involutions have trace ± 8 on V_{\min} , we would need a trace of ± 8 from the remaining factors of $V_{\min} \downarrow_L$, a module of dimension 6, so not possible. Thus this is not a summand of $V_{\min} \downarrow_L$.

For $p = 19$, the module of dimension congruent to 6 modulo p has socle structure

$$5, 7, 9, 11, 13, 15/5, 7, 9, 11, 13, 15$$

and has dimension 120, so cannot be a summand of $V_{\min} \downarrow_L$.

For $p = 23$ the module of dimension congruent to 10 modulo p has socle structure

$$3, 5, 7, 9, 11, 13, 15, 17, 19, 21/3, 5, 7, 9, 11, 13, 15, 17, 19, 21$$

and has dimension 240, so cannot be a summand of $V_{\min} \downarrow_L$.

We now list the unipotent classes of interest using [13, Table 7].

- (i) $A_3 + A_2$, $p = 5$, acting as $5^6, 4^2, 3^4, 2^2, 1^2$;

- (ii) $A_4, p = 5$, acting as $5^{10}, 1^6$;
- (iii) $A_4 + A_1, p = 5$, acting as $5^{10}, 2^2, 1^2$;
- (iv) $A_4 + A_2, p = 5$, acting as $5^{10}, 3^2$;
- (v) $(A_5)'', p = 7$, acting as $7^2, 6^7$;
- (vi) $D_4 + A_1, p = 7$, acting as $7^6, 2^5, 1^4$;
- (vii) $D_5(a_1), p = 7$, acting as $7^6, 3^2, 2^2, 1^4$;
- (viii) $(A_5)', p = 7$, acting as $7^4, 6^4, 1^4$;
- (ix) $A_5 + A_1, p = 7$, acting as $7^4, 6^3, 5^2$;
- (x) $D_5(a_1) + A_1, p = 7$, acting as $7^6, 4, 2^5$;
- (xi) $D_6(a_2), p = 7$, acting as $7^6, 5^2, 4$;
- (xii) $E_6(a_3), p = 7$, acting as $7^6, 5^2, 1^4$;
- (xiii) $E_7(a_5), p = 7$, acting as $7^6, 6, 4^2$;
- (xiv) $A_6, p = 7$, acting as 7^8 ;
- (xv) $D_6, p = 11$, acting as $11^4, 10, 1^2$;
- (xvi) $E_6(a_1), p = 11$, acting as $11^4, 5^2, 1^2$;
- (xvii) $E_7(a_3), p = 11$, acting as $11^4, 10, 2$;
- (xviii) $E_6, p = 13$, acting as $13^4, 1^4$;
- (xix) $E_7, p = 19$, acting as $19^2, 18$.

We start with $p = 5$, proving that there are always at least six odd-dimensional summands so that we can assume that $a = 1, 2$ when doing the hard work.

Proposition 11.3 Let $p = 5$ and $a \geq 1$.

- (i) If $a = 1, 2$ then H fixes a line on either V_{\min} or $L(G)$.
- (ii) If $a \geq 3$ then H is a blueprint for V_{\min} .

Proof: The traces on elements of orders 2 and 3 yield conspicuous sets of composition factors for $L = \text{PSL}_2(5)$ of

$$3^{12}, 1^{20}, \quad 5^2, 3^{14}, 1^4, \quad 5^9, 3^3, 1^2, \quad 5^6, 3^6, 1^8.$$

As we have seen, only $P(3) = 3/1, 3/3$ has a trivial composition factor and no trivial submodule or quotient, and so the first, third and fourth cases all fix lines on V_{\min} . However, in the third case this means that L cannot embed with these factors: as it fixes a line on V_{\min} , from Lemma 2.5 we see that L lies in either an E_6 -parabolic, with factors $1, 1, 27, 27^*$ or a B_5 -subgroup, factors $1, 1, 11^2, 32$, neither of which is compatible with $5^9, 3^3, 1^2$, so this case cannot occur.

For the remaining case of $5^2, 3^{14}, 1^4$, we switch to the Lie algebra. There are two possibilities for the corresponding composition factors of $L(G)$ (since an element of order 3 with trace 2 on V_{\min} can have trace either -2 or 7 on $L(G)$) are

$$5^{10}, 3^{22}, 1^{17}, \quad \text{and} \quad 5^{13}, 3^{19}, 1^{11},$$

both of which must have trivial submodules as again we can only cover a 1 by $3/1, 3/3$. This proves (i).

Recall from Lemma 5.24 that the only simple modules for H with non-trivial 1-cohomology when $a \geq 2$ have dimension 8 and restrict to L as $5 \oplus 3$ by Lemma 5.21. When $a = 2$, if $V_{\min} \downarrow_L$ has composition factors $3^{12}, 1^{20}$ then $V_{\min} \downarrow_H$ has at least eight trivial composition factors and no factors of dimension 8, so $V_{\min} \downarrow_H$ has eight trivial summands. Similarly, if the composition factors of $V_{\min} \downarrow_L$ are $5^6, 3^6, 1^8$ then $V_{\min} \downarrow_H$ must have at least two trivial composition factors by Lemma 5.21, and for every composition factor of dimension 8 we must have another trivial factor, so $V_{\min} \downarrow_H$ always has pressure at most -2 , so (ii) holds.

We thus may assume that $V_{\min} \downarrow_L$ has factors $5^2, 3^{14}, 1^4$. We have at most two 8s in $V_{\min} \downarrow_H$ since there are only two 5s in $V_{\min} \downarrow_L$, and hence there can be at most a single trivial composition factor in $V_{\min} \downarrow_H$, else H fixes a line on V_{\min} . We thus get two cases: there is a trivial composition factor and there is not.

If there is a trivial factor we have $8^2, 1$ in $V_{\min} \downarrow_H$, and the remaining factors of $V_{\min} \downarrow_H$ restrict to L as $3^{12}, 1^3$, so we need factors $8^2, 4^3, 3^9, 1$. For $a = 2$, there is no such set of composition factors, so we cannot have a trivial composition factor in $V_{\min} \downarrow_H$.

For $a = 2$ there are up to field automorphism five possible sets of composition factors, which are

$$5_1^2, 4^4, 3_1^{10}, \quad 8_{2,1}, 5_1, 4^4, 3_1^9, \quad 8_{2,1}^2, 4^4, 3_1^7, 3_2, \quad 8_{2,1}^2, 4^4, 3_1^6, 3_2^2, \quad 15_{2,1}^2, 4^2, 3_1^4, 3_2^2.$$

The fact that 4 has an extension with 3_1 and 3_2 (see Lemma 5.25) makes deducing the module structure difficult, and so we turn to the Lie algebra in all cases. These are

$$\begin{aligned} &8_{2,1}^3, 5_1^{10}, 4_{1,2}^5, 3_1^{11}, 1^6, \quad 8_{2,1}^5, 5_1^5, 4_{1,2}^6, 3_1^{10}, 3_2, 1^{11}, \quad 9, 8_{2,1}^7, 5_1^3, 3_1^5, 3_2^3, 1^7, \\ &9^3, 8_{2,1}^4, 5_1^3, 4^8, 3_1^6, 3_2, 1^6, \quad 15_{2,1}^2, 9_{1,2}^4, 8_{1,2}^2, 8_{2,1}, 5_1^3, 5_2, 4_{1,2}^3, 3_1^2, 3_2, 1^2 : \end{aligned}$$

each of these has non-positive pressure, as needed. (Remember that $8_{2,1}$ has 1-cohomology but $8_{1,2}$ does not by Lemma 5.24.)

Finally, suppose that $a \geq 3$. From Lemma 5.21 we see the following facts: firstly, in any even-dimensional composition factor of $V_{\min} \downarrow_H$ there are the same number of 3s as 5s and 1s combined on restriction to L , and secondly, in any odd-dimensional factor of $V_{\min} \downarrow_H$ there is at most one more 5 and 1 combined than 3 on restriction to L . This means that if $V_{\min} \downarrow_L$ has factors $5^6, 3^6, 1^8$, there must be at least six odd-dimensional composition factors. Lemma 5.21 easily shows that if the factors of $V_{\min} \downarrow_L$ are $3^{12}, 1^{20}$ then there must be at least eight trivial factors in $V_{\min} \downarrow_H$, and if we have $5^2, 3^{14}, 1^4$ then we have at least six 3s in $V_{\min} \downarrow_H$, so in all cases we have at least six odd-dimensional composition factors, so an element of order $(p^a \pm 1)/2 > 30$ has a 1-eigenspace of dimension at least 6. This means that H is a blueprint by Proposition 4.10, as needed for the proposition. \square

Having completed $p = 5$, we now move on to $p = 7$. This time $p^a = 7, 49$ will need to be considered, but $p^a = 343$ is above $2 \cdot v(E_7) = 150$.

Proposition 11.4 Suppose that $p = 7$.

(i) If $a = 1$ then either H fixes a line on V_{\min} or $L(G)$, or the actions of H on V_{\min} and $L(G)$ are

$$7^{\oplus 4} \oplus P(3)^{\oplus 2} \quad \text{and} \quad 7^{\oplus 5} \oplus P(5)^{\oplus 6} \oplus P(3).$$

(ii) If $a = 2$ then either H is a blueprint for V_{\min} or H fixes a line on V_{\min} .

Proof: We first compute the possible composition factors of $V_{\min} \downarrow_H$ when $a = 1$, using the traces of elements of orders 2, 3 and 4. There are seven of these, given by

$$3^{12}, 1^{20}, \quad 5^2, 3^{14}, 1^4, \quad 5^6, 3^6, 1^8, \quad 7, 5^9, 3, 1, \quad 7^2, 5^6, 3^2, 1^6, \quad 7^4, 5^2, 3^6, \quad 7^6, 1^{14}.$$

As in the case of $p = 5$, the only indecomposable module with a trivial composition factor but no trivial submodule or quotient is $P(5) = 5/1, 3/5$, so we need twice as many 5s as 1s and as many 3s as 1s. Thus all but the fourth and sixth cases must fix lines on V_{\min} .

Case 4: If the factors are $7, 5^9, 3, 1$, then H cannot fix a line or hyperplane on V_{\min} , since by Lemma 2.5 the line stabilizers for V_{\min} are contained in either an E_6 -parabolic – composition factors $27, 27^*, 1^2$ – or a subgroup $q^{1+32}B_5(q) \cdot (q-1)$, – composition factors $32, 11^2, 1^2$ – neither of which can work. As there are no self-extensions of the 5, $V_{\min} \downarrow_H$ is

$$7 \oplus P(5) \oplus 5^{\oplus 7},$$

with u acting as $7^3, 5^7$, not in [13, Table 7], so there does not exist an embedding of H into G with these factors.

Case 6: We are left with $7^4, 5^2, 3^6$. Here, the lack of trivials means that the possible summands are

$$3/5 \oplus 5/3, \quad 3/3 \quad 3/3, 5/3, \quad 3/3, 5 \oplus 3, 5/3, \quad 3, 5/3, 5.$$

Therefore we need an even number of 1s, 4s and 7s in the Jordan block structure of u , with at least four more 7s than the 1s, 2s and 4s combined. Examining the list above, we see only two examples of this, namely (xiii) and (xiv). this yields the two possible embeddings $V_{\min} \downarrow_H$ to be

$$7^{\oplus 4} \oplus 3/3 \oplus 3/3, 5 \oplus 3, 5/3, \quad \text{and} \quad 7^{\oplus 4} \oplus P(3)^{\oplus 2}.$$

The traces of semisimple elements of orders 3 and 4 on V_{\min} yield two possibilities each for the semisimple class, and so we get four possible sets of composition factors for $L(G) \downarrow_H$, namely

$$7^6, 5^{10}, 3^{10}, 1^{11}, \quad 7^8, 5^{10}, 3^6, 1^9, \quad 7^3, 5^{13}, 3^{13}, 1^8, \quad 7^5, 5^{13}, 3^9, 1^6.$$

Apart from the last one, each of these has enough trivials and not enough 5s to ensure that H fixes a line on $L(G)$, since $P(5) = 5/1, 3/5$ is the only indecomposable module for H with a trivial factor but no trivial submodule or quotient.

We now remove the first possible action on V_{\min} , using the simple fact that for $p \geq 5$, the symmetric square of V_{\min} is the sum of $L(G)$ and the 1463-dimensional module $L(2\lambda_1)$, Lemma 2.1. The symmetric square of the first module is a sum of projectives and

$$(3, 5/1, 3, 5 \oplus 1, 3, 5/3, 5)^{\oplus 2} \oplus 5 \oplus 3 \oplus 1.$$

Since u comes from class $E_7(a_5)$, and this acts on $L(G)$ as $7^{17}, 5, 3^3$, we must have two of the summands of dimension 17 in $L(G) \downarrow_H$, hence H fixes a line on $L(G)$. This completes the proof for $a = 1$.

Now let $a = 2$, so that $H = \text{PSL}_2(49)$, and recall that $L \leq H$ is a copy of $\text{PSL}_2(7)$. At the start of this proof we gave the conspicuous sets of composition factors for $V_{\min} \downarrow_L$, and from Lemmas 5.23 and 5.24 we see that the only simple modules for H with non-trivial 1-cohomology have dimension 12 and restrict to L as $7 \oplus 5$, and only the trivial module for H restricts to L with more 1s than 3s. These two facts mean that if $V_{\min} \downarrow_L$ has factors the first, third, fifth and seventh cases then $V_{\min} \downarrow_H$ has trivial composition factors, and these are summands except for the fifth case, and there we have at least four trivials and at most two 12s, so pressure at most -2 . The fourth case cannot occur, as we proved, so $V_{\min} \downarrow_L$ has factors either $5^2, 3^{14}, 1^4$, and $V_{\min} \downarrow_H$ can have no trivial factors else it has a trivial summand, or $7^4, 5^2, 3^6$.

In the case of $5^2, 3^{14}, 1^4$, from Lemma 5.23, apart from 3, there are no simple modules for H whose restriction to L has more 3s than other factors, and the composition factors of $V_{\min} \downarrow_H$ have dimensions 1, 3, 4, 5, 8 and 9. In particular, this means that $V_{\min} \downarrow_H$ has at least eight 3-dimensional composition factors. By Lemma 5.26, of these modules only 8s can have an extension with 3s, with there being at most two of those, so the 3-pressure is at least 6. This means that $V_{\min} \downarrow_H$ has at least four 3-dimensional summands, so the action of the unipotent element u on V_{\min} has at least four Jordan blocks of size 3. There are no non-generic unipotent classes with this property, as we saw in the list at the start of this section, and so H is a blueprint for V_{\min} .

Thus we end with $V_{\min} \downarrow_L$ being $7^4, 5^2, 3^6$, and Lemma 5.23 implies that H has at least two 7s and four 3s on V_{\min} . The remaining composition factors have dimension 3, 5, 7, 8, 12 or 15: using the traces of semisimple elements, we find exactly four conspicuous sets of composition factors for $V_{\min} \downarrow_H$ with the correct restriction to L , and for each of these the eigenvalues of an element of order 24 determine its conjugacy class, and this lies inside F_4 , hence a blueprint by Lemma 4.9. Thus H is a blueprint for V_{\min} , as needed. \square

Proposition 11.5 Suppose that $p = 11$.

- (i) If $a = 1$ then either H is a blueprint for V_{\min} or H fixes a line on V_{\min} .
- (ii) If $a = 2$ then H is a blueprint for V_{\min} .

Proof: Let $a = 1$ firstly, and suppose that H is not a blueprint for V_{\min} , so in particular the class to which the unipotent u belongs is non-generic, thus cases (xv) to (xvii) from the start of this section. In each case we either have 5^2 or 10 in the action of u . A single block of size 10 (as V_{\min} is self-dual) must come from $5/5$, and for the 5^2 the indecomposable modules of dimension congruent to 5 modulo 11 are 5 itself, $5, 7/3, 5, 7$ and its dual of dimension 27, and $3, 5, 7, 9/1, 3, 5, 7, 9$ and its dual, of dimension 49. Thus if we are in case (xvi), so u belongs to class $E_6(a_1)$ acting as $11^4, 5^2, 1^2$, we therefore have $5^{\oplus 2}$ as a summand of $V_{\min} \downarrow_H$, or

$$V_{\min} \downarrow_H = 5, 7/3, 5, 7 \oplus 3, 5, 7/5, 7 \oplus 1^{\oplus 2};$$

an involution $x \in H$ acts with trace 0 on this module, so it is not allowed. Therefore in all cases $V_{\min} \downarrow_H$ has either $5/5$ or $5^{\oplus 2}$ as a summand, contributing 2 to the trace of x .

If u comes from class $E_7(a_3)$, acting as $11^4, 10, 2$, the self-dual module that can contribute 2 to the action of u is $5, 7/5, 7$, so that $V_{\min} \downarrow_H$ has composition factors at least $7^2, 5^4$. The only conspicuous sets of composition factors with this many 7s and 5s are

$$9^2, 7^2, 5^4, 1^4 \quad \text{and} \quad 7^2, 5^6, 3^2, 1^6,$$

with the latter being incompatible with the unipotent action and the former implying that $V_{\min} \downarrow_H$ is $P(1)^{\oplus 2} \oplus 5/5 \oplus 5, 7/5, 7$. Clearly therefore H fixes a line on V_{\min} .

For D_6 and $E_6(a_1)$ we already have the composition factors of 5^2 , and the Jordan blocks 1^2 in the action of u come either from two trivial summands, so we are done, or come from $5/7$ or $3/9$ and their duals, on each of which x acts with trace 0. From the previous case we know that $5/7$ yields

$$P(1)^{\oplus 2} \oplus 5/7 \oplus 7/5 \oplus M,$$

where M is either $5^{\oplus 2}$ or $5/5$, and again we fix two lines on V_{\min} . If we have $3/9 \oplus 9/3$, we again have a unique conspicuous set of composition factors, and this time it is the very similar

$$P(1)^{\oplus 2} \oplus 3/9 \oplus 9/3 \oplus M,$$

where M is as before, proving the result.

Now let $a = 2$, and suppose that H is not a blueprint for V_{\min} . In particular, by Proposition 4.10 there are at most four odd-dimensional composition factors in $V_{\min} \downarrow_H$. When the composition factors of $V_{\min} \downarrow_L$ are $9^2, 7^2, 5^4, 1^4$, as they are in two cases above, all of the 1s become trivial composition factors for H , as there are no 3s in $V_{\min} \downarrow_L$. Also, because there are no 11s either, there can be no simple modules in $V_{\min} \downarrow_H$ with non-trivial 1-cohomology by Lemma 5.24, as a 20-dimensional module restricts as $11 \oplus 9$. Thus the potential embeddings of L into G with these factors, where L has a trivial submodule but no trivial summand, cannot be extended to embeddings of H into G . The same statement holds when the composition factors are $9^4, 5^2, 3^2, 1^4$, as we have at least two trivial composition factors.

Thus we may assume that $V_{\min} \downarrow_L$ has two trivial summands, along with the two 5s, with the rest being projective. The conspicuous such modules are

$$11^{\oplus 2} \oplus P(1)^{\oplus 2} \oplus 1^{\oplus 2} \oplus M \quad \text{and} \quad P(9)^{\oplus 2} \oplus 1^{\oplus 2} \oplus M.$$

The first case has six trivial factors and no 3s, so $V_{\min} \downarrow_H$ has six trivial factors and hence H is a blueprint for V_{\min} . In the second case there are two more trivials than 3s so $V_{\min} \downarrow_H$ has two trivial factors, and $V_{\min} \downarrow_L$ has four 9s and no 7s or 11s, so $V_{\min} \downarrow_H$ has four 9s, thus H has six odd-dimensional factors on V_{\min} and so is a blueprint for V_{\min} , as needed. \square

For $p = 13$, since $169 > 150 = 2 \cdot v(E_7)$, we need only consider $a = 1$.

Lemma 11.6 Suppose that $p = 13$ and $a = 1$. Then either H is a blueprint for V_{\min} or H fixes a line on either V_{\min} or $L(G)$.

Proof: If H is not a blueprint for V_{\min} then u acts non-generically, and so we are in case (xviii) of the list of possible unipotent actions at the start of this subsection. Suppose that H has no trivial summand on V_{\min} , so that the 1^4 in the action of u all comes from indecomposables of dimension 14; this means that there are eight composition factors of dimensions between 3 and 11, as indecomposables of dimension 14 have the form $i/(14-i)$. Since the trace of an involution x is ± 8 , and the trace of x on $i/(14-i)$ is ± 2 , each of them has the same trace. Adding in the traces of elements of orders 3 and 4, the unique such set of composition factors is $11^2, 7^4, 3^2$. This gives $V_{\min} \downarrow_H$ as

$$11/3 \oplus 3/11 \oplus 7/7 \oplus 7/7.$$

Using these traces, we can determine two possibilities for the action of L on the adjoint module $L(G)$: these are $13^3, 11, 9^3, 7^5, 5^3, 3, 1^3$ and $13^2, 11^3, 9^4, 7, 5^4, 3^3, 1^2$. Either way, H fixes a line on $L(G)$, since the only module with a trivial factor but not a trivial submodule or quotient is $P(11) = 11/1, 3/11$. The proof is complete. \square

The last case is $p = 19$, where again we only have $a = 1$.

Proposition 11.7 Suppose that $p = 19$ and $a = 1$. If H is not a blueprint for V_{\min} then H centralizes a 2-space on V_{\min} and is a non- G -cr subgroup of the E_6 -parabolic acting on V_{\min} as

$$P(1)^{\oplus 2} \oplus 9/9.$$

Proof: If H is not a blueprint for V_{\min} then in particular u is non-generic, and so we are in case (xix) from the list at the start of the section, i.e., u is regular and acts as $19^2, 18$. We need a self-dual indecomposable module of dimension congruent to 18 modulo 19, and there is only one of these by Lemma 5.17, namely $9/9$, and the remainder of the module is projective. If x denotes an involution in H then x has trace ± 8 on V_{\min} , and has trace 2 on $9/9$, leaving a trace of 6 or -10 on the remaining projective summand. The trace of x on $P(i)$ for $3 \leq i \leq 17$ is ± 2 , the trace of x on 19 is -1 , and on $P(1)$ it is 3. Thus $V_{\min} \downarrow_H$ is $P(1)^{\oplus 2} \oplus 9/9$, as needed. \square

This non- G -cr subgroup was constructed at the end of Section 9.

12 E_7 in odd characteristic: SL_2 embedding

In this section, k is a field of characteristic $p \geq 3$ and $G = E_7(k)$, by which we mean the simply connected form, i.e., $|Z(G)| = 2$ and $G' = G$. Let \bar{G} be an almost simple group with socle $G/Z(G)$. From [10] we see that $v(E_7) = 75$ for odd integers, so if H is any subgroup of G with a semisimple element of odd order 77 or more, then H is a blueprint for V_{\min} , with the same holding for $N_{\bar{G}}(H)$. In addition, in [9] we prove that $\mathrm{SL}_2(9)$ cannot yield a maximal subgroup of \bar{G} either, so here we let $H = \mathrm{SL}_2(p^a)$ with $a = 3, 4$ if $p = 3$ and $p^a \leq 150 = 2 \cdot v(E_7)$ if $p \geq 5$. In order for H not to be contained in a centralizer of a non-central involution, we require that $Z(G) = Z(H)$. Let $L = \mathrm{SL}_2(p) \leq H$ and let u denote a unipotent element of L of order p .

On V_{\min} , since we consider $\mathrm{SL}_2(p^a)$ rather than $\mathrm{PSL}_2(p^a)$, there can be no trivial composition factors in $V_{\min} \downarrow_H$, but rather 2-dimensional factors. We will thus normally aim to show that H is a blueprint for V_{\min} , that H fixes a line on $L(G)$, or that H fixes a 2-space on V_{\min} .

12.1 Characteristic 3

We let $p = 3$ and $a = 3, 4$. We begin with the case where the action of $H = \mathrm{SL}_2(3^a)$ on V_{\min} is definable over a subfield of \mathbb{F}_{3^a} .

Proposition 12.1 Suppose that $p = 3$, that $a = 3, 4$, and that the composition factors of $V_{\min} \downarrow_H$ are invariant under a non-trivial field automorphism of H .

- (i) If $a = 3$ then the composition factors of $V_{\min} \downarrow_H$ are either

$$8, (2_1, 2_2, 2_3)^8, \quad \text{or} \quad 8^4, (2_1, 2_2, 2_3)^4;$$

in the first case H fixes a line on $L(G)$, and in the second case $N_{\bar{G}}(H)$ is contained in a member of \mathcal{X}^σ .

- (ii) The case $a = 4$ cannot occur.

Proof: For $a = 4$, it turns out that the traces of semisimple elements of orders 5 and 8 are enough to eliminate all possible sets of composition factors for $V_{\min} \downarrow_H$, so we concentrate on the case $a = 3$.

The traces of semisimple elements of H are enough to confirm the first part of this statement. If the composition factors of $V_{\min} \downarrow_H$ are $8, (2_1, 2_2, 2_3)^8$ then the composition factors of H on $L(G)$ are

$$(4_{1,2}, 4_{2,3}, 4_{1,3})^8, (3_1, 3_2, 3_3), 1^{28},$$

which has pressure -4 and so H fixes a line on $L(G)$. Suppose therefore that the composition factors of $V_{\min} \downarrow_H$ are $8^4, (2_1, 2_2, 2_3)^4$, and to start that there is an 8 in the socle of $V_{\min} \downarrow_H$. Let y be the diagonal matrix with entries ζ and ζ^{-1} , where ζ is a primitive 26th root of unity. The eigenvalues of y^2 on 8 are

$$1^2, \zeta^{\pm 4}, \zeta^{\pm 10}, \zeta^{\pm 12},$$

and so we look for elements of order 26 in G that square to y^2 and stabilize these eigenspaces of the action of y^2 on V_{\min} : in fact, there are elements of G lying in seven distinct conjugacy classes of semisimple elements, one of which contains y itself, and all of which have no (-1) -eigenspace on $L(G)$. Any of these can be used with Corollary 4.14, but taken together they show that the stabilizer of an 8 that is a submodule of $V_{\min} \downarrow_H$ must be very large, and in particular positive dimensional.

Thus 8 is not a submodule of $V_{\min} \downarrow_H$. If a 2_i submodule is stabilized by $N_{\bar{G}}(H)$ and \bar{G} then H is contained inside a member of \mathcal{X}^σ , so we may assume that this is not the case, either because $N_{\bar{G}}(H)$ induces the field automorphism on H or that no 2_i is σ -stable (because k does not contain \mathbb{F}_{27} for example). In this case there must be an $N_{\bar{G}}(H)$ -stable H -submodule W isomorphic to $2_1 \oplus 2_2 \oplus 2_3$, and the eigenvalues of y^2 on W are $\zeta^{\pm 2}$, $\zeta^{\pm 6}$ and $\zeta^{\pm 8}$. In this case we do the same as for 8, finding again seven distinct classes containing elements that square to y^2 and preserve these eigenspaces. Thus $N_{\bar{G}}(H)$ is always contained in a member of \mathcal{X}^σ as needed. \square

We may now attack the case of $a = 3$. We will not prove anything specific about the embeddings of H into G , except that they are always contained in positive-dimensional subgroups.

Proposition 12.2 Suppose that $p = 3$ and $a = 3$. The subgroup $N_{\bar{G}}(H)$ is contained inside a member of \mathcal{X}^σ .

Proof: There are 284 conspicuous sets of composition factors for $V_{\min} \downarrow_H$, but only 137 of these have corresponding sets of factors on $L(G)$, each of these being unique. Of these, seventy-seven have either no 2_i or positive 2_i -pressure for each $i = 1, 2, 3$, and of these only sixty-seven have either no trivial or positive pressure on $L(G)$. One of these is invariant under the field automorphism, which we saw earlier, so we are left with twenty-two sets of composition factors, up to field automorphism.

The module 2_i has non-split extensions only with $2_{i\pm 1}$, $6_{i-1,i}$ and 8, so if there are 2s but no $6_{i,i-1}$ or 8 appearing with multiplicity 2 or above, then H fixes a 2-space on V_{\min} : two sets of composition factors (up to field automorphism) satisfy this, so we are down to twenty sets of composition factors.

There are eighteen (six up to field automorphism) sets of composition factors with no 8s, and since 2_i only has extensions with the 8, $2_{i\pm 1}$ and $6_{i-1,i}$, and of course there can be no 2s in the socle of $V_{\min} \downarrow_H$, the socle must consist of $6_{i-1,i}$ s, plus modules we can remove without exposing 2s. In each case there is a unique i such that $6_{i-1,i}$ appears with multiplicity at least 2, so this must be the socle and all 2s must be stacked on top of it in some way. In each case we cannot place enough 2s on top of each $6_{i-1,i}$, and so we must fix a 2-space in all these cases. This reduces us to fourteen sets of composition factors.

Thus we have at least one 8 in $V_{\min} \downarrow_H$. There are five conspicuous sets of composition factors with a single 8, up to field automorphism, namely

$$\begin{aligned} &8, 6_{2,1}^2, 6_{3,1}^2, 2_1^6, 2_2^4, 2_3^2, & 8, 6_{1,3}^2, 6_{2,3}, 6_{3,1}^2, 2_1^5, 2_2^2, 2_3^2, & 8, 6_{3,1}^5, 2_1^5, 2_2^2, 2_3^2, \\ &8, 6_{1,2}^2, 6_{2,1}^2, 6_{3,1}, 6_{3,2}, 2_1^2, 2_2^2, 2_3^2, & 18_{2,3,1}, 8, 6_{2,1}, 6_{3,1}^2, 2_1^3, 2_2, 2_3^2, \end{aligned}$$

Since there is a single 8, we must still have the $6_{i-1,i}$ s in the socle, and so again we can eliminate these cases as follows: each of these only has a single $6_{i-1,i}$ appearing with a non-unital multiplicity, and it appears with multiplicity 2, so this appears once in the socle and all 2s lie above this, so we consider the $\{2, 6, 8\}$ -radical of $P(6_{1,2})$, and then remove all quotients that are not $6_{1,2}$, since V_{\min} is self-dual. This module is

$$6_{1,2}/2_2/2_3/2_2/6_{1,2},$$

and so we can only support three 2s, but there are clearly far too many. This is enough to complete the first four cases, but the fifth has an 18. We take the appropriate radical of $P(6_{3,1})$, and again remove all composition factors from the top that are not $6_{3,1}$, to get the module

$$6_{3,1}/2_1/2_2, 6_{3,1}/2_1, 6_{2,1}, 18_{2,3,1}/6_{3,1},$$

which does not have a 2_3 in it. Thus all these conspicuous sets of composition factors yield a stabilized 2-space.

Thus we are down to nine, which are below.

$$\begin{aligned} &8^2, 6_{2,1}^2, 6_{1,2}, 6_{3,1}, 2_1^4, 2_2^3, 2_3, & 8^2, 6_{2,1}, 6_{1,3}, 6_{3,1}^2, 2_1^4, 2_2^2, 2_3^2, & 8^2, 6_{3,2}^2, 6_{1,2}, 6_{2,1}, 6_{3,1}, 6_{1,3}, 2_1, 2_2, \\ &8^2, 6_{3,2}^2, 6_{2,1}, 6_{3,1}^2, 2_1^2, 2_2^2, 2_3, & 8^2, 6_{1,3}, 6_{3,1}, 6_{2,3}^2, 6_{1,2}, 2_1^2, 2_2^2, 2_3, & 8^2, 6_{3,2}, 6_{2,1}, 6_{2,3}, 6_{1,3}^2, 2_1^2, 2_2^2, 2_3, \\ &18_{1,2,3}, 8^2, 6_{2,1}, 6_{3,1}, 2_1^2, 2_2^2, 2_3, & 18_{1,2,3}, 8^3, 6_{1,3}, 2_1^3, 2_2, & 18_{1,2,3}, 8^2, 6_{2,1}, 6_{1,3}, 2_1^3, 2_2^2. \end{aligned}$$

Recall that 2_i has extensions with $2_{i\pm 1}$, $6_{i-1,i}$ and 8, and no other simple modules.

Cases 1,6,7,8,9: In the first, sixth, seventh, eighth and ninth cases, $6_{i-1,i}$ does not occur with multiplicity greater than 1 for all i , and so if it occurs in the socle then it is a summand. Therefore, for these six sets of composition factors, we can remove all quotients and submodules from $V_{\min} \downarrow_H$ other than 8, and yield a submodule W of $P(8)$, which contains all composition factors of dimension 2 in $V_{\min} \downarrow_H$.

The $\{2_i, 6_{i,i+1}, 6_{i+1,i}, 18_{1,2,3}\}$ -radical of $P(8)/\text{soc}(P(8))$ is

$$2_1, 2_2, 2_3, 6_{1,2}, 6_{2,3}, 6_{3,1}/2_1, 2_2, 2_3, 6_{2,1}, 6_{3,2}, 6_{1,3}/8,$$

and so if W has two 8s and three 2_i s then H must stabilize a 2-space on V_{\min} , eliminating the first and ninth cases immediately. It also means that W has at most four socle layers except for the eighth case, which we eliminate easily, since the $\{2_1, 2_2, 6_{1,3}, 8, 18_{1,2,3}\}$ -radical of $P(8)$ is

$$8/2_1, 2_2, 6_{1,3}/8,$$

which doesn't have enough 2_1 s, so H fixes a 2-space of V_{\min} .

Since W has at most four socle layers and is self-dual, none of the $6_{i,i+1}$ can occur in W . Also, any 2_i that occurs with multiplicity 1 in $V_{\min} \downarrow_H$ must occur in the second socle layer, and cannot have any extensions with a $2_{i\pm 1}$ in the third socle layer. However, both $2_{i\pm 1}$ in the third layer have an extension with the 2_i in the second layer, so cannot exist in W . In other words, we cannot have two 2_i in W , eliminating the sixth and seventh cases.

Case 5: Remove all composition factors from the top and bottom of $V_{\min} \downarrow_H$ to yield a module W with socle a submodule of $8 \oplus 6_{2,3}$ and all 2s in it.

The preimages of the $\{2_1, 2_2, 2_3, 6_{1,2}, 6_{1,3}, 6_{2,3}\}$ -radical of the module $P(8)/\text{soc}(P(8))$ and $\{2_1, 2_2, 2_3, 6_{1,2}, 6_{1,3}, 8\}$ -radical of $P(6_{2,3})/\text{soc}(P(6_{2,3}))$ are

$$2_1, 2_2, 2_3, 6_{2,3}/2_1, 2_2, 2_3, 6_{1,3}/8 \quad 2_2, 2_3/2_1, 8/2_3, 6_{1,3}/6_{2,3},$$

and the fact that there is a single 2_3 means it must lie in the second socle, and has no extensions with the other 2_i composition factors. But then all other 2_i s lie in the second socle layer as well, and that means we cannot fit enough 2s in, so H fixes a 2-space on V_{\min} .

Cases 2,3,4: Here we wish to apply Corollary 4.14, so let y be the diagonal matrix with entries ζ^2, ζ^{-2} for ζ a primitive 26th root of unity, so that y has order 13 and acts with eigenvalues $\zeta^{\pm 2}$ on 2_1 . The eigenvalues of y^2 on $6_{3,1}$ and 8 are

$$\zeta^{\pm 4}, \zeta^{\pm 8}, \zeta^{\pm 12} \quad \text{and} \quad 1^2, \zeta^{\pm 4}, \zeta^{\pm 10}, \zeta^{\pm 12}.$$

If we can find an element \hat{y} of order 26 in $G \setminus H$ that has no (-1) -eigenspace on $L(G)$ and that stabilizes a submodule W of $V_{\min} \downarrow_H$, then by Corollary 4.14 the subgroup $\langle H, \hat{y} \rangle$, which cannot be all of G , is not

almost simple modulo $Z(G)$ either, and so by Proposition 3.3 the stabilizer of W , and any submodule of $V_{\min} \downarrow_H$ isomorphic to W , is contained in a member of \mathcal{X}^σ , and so H and indeed $N_{\bar{G}}(H)$ are contained inside a member of \mathcal{X}^σ .

In the third case, the composition factor $6_{3,1}$ only has extensions with 2_1 and $6_{2,1}$ from the composition factors of $V_{\min} \downarrow_H$, so this must split off as a summand. In the second and fourth cases, if 8 is not a submodule of $V_{\min} \downarrow_H$ and $V_{\min} \downarrow_H$ does not a 2-dimensional submodule then $V_{\min} \downarrow_H$ is a submodule of $P(6_{3,1})$ and $P(6_{3,2} \oplus 6_{3,1})$ respectively. The $\text{cf}(V_{\min} \downarrow_H)$ -radical of $P(6_{3,1})$ in case 2 is

$$6_{3,1}/2_1, 2_3/2_2, 8/2_1, 6_{2,1}/6_{3,1},$$

so 8 must be a submodule of $V_{\min} \downarrow_H$. Similarly, the $\text{cf}(V_{\min} \downarrow_H)$ -radicals of $P(6_{3,1})$ and $P(6_{3,2})$ in case 4 is

$$6_{2,1}/6_{3,1}, 8/2_1, 2_3, 6_{3,2}/2_2, 8/2_1, 6_{2,1}/6_{3,1} \quad \text{and} \quad 6_{3,1}/2_1, 6_{2,1}/8/6_{3,2},$$

so since there is no 2_2^2 in this, 8 must be in the socle of $V_{\min} \downarrow_H$ again.

We have therefore proved that in cases 2 and 4, 8 must be a submodule, and in case 3, $6_{3,1}$ must be a submodule. We therefore find, in each case, an element \hat{y} in $G \setminus H$ of order 26, squaring to y^2 , and stabilizing the eigenspaces of the particular stabilized submodule. On V_{\min} these have eigenvalues

$$\begin{aligned} &1^4, (-\zeta^{\pm 1})^6, (\zeta^{\pm 2})^4, (\zeta^{\pm 3}), (-\zeta^{\pm 3})^2, (\zeta^{\pm 4})^2, (-\zeta^{\pm 4})^3, (\zeta^{\pm 5})^3, (-\zeta^{\pm 6})^5, \\ &1^4, (\zeta^{\pm 1})^2, (-\zeta^{\pm 1})^2, (-\zeta^{\pm 2})^5, (\zeta^{\pm 3})^3, (-\zeta^{\pm 3}), (\zeta^{\pm 4})^4, (\zeta^{\pm 5}), (-\zeta^{\pm 5})^3, (-\zeta^{\pm 6})^5, \\ &1^4, (\zeta^{\pm 1})^2, (-\zeta^{\pm 1}), (-\zeta^{\pm 2})^6, (\zeta^{\pm 3})^4, (-\zeta^{\pm 3}), (\zeta^{\pm 4})^5, (-\zeta^{\pm 5})^3, (-\zeta^{\pm 6})^4. \end{aligned}$$

Thus the result holds. \square

Proposition 12.3 Suppose that $p = 3$ and $a = 4$. Either H is a blueprint for V_{\min} or the composition factors of $V_{\min} \downarrow_H$ are

$$18_{4,2,3}, 8_{1,2,3}^2, 8_{1,2,4}, 6_{1,3}, 6_{4,1}, 2_1,$$

and $N_{\bar{G}}(H)$ lies inside an element of \mathcal{X}^σ .

Proof: Using semisimple elements of order up to 41, one whittles down the 55 million or so possible sets of composition factors for a module of dimension 56 to just 190 up to field automorphism, of which two fail the trace of an element of order 80, leaving 188.

Of these, we consider an element y of order 41 in H , and whether there exists an element of order 123 in G cubing to y and stabilizing the same subspaces of V_{\min} . If this is true then y is a blueprint for V_{\min} since $123 > v(E_7)$. Indeed, a computer check shows that this is true for 187 of the 188 semisimple elements involved. The remaining one comes from the conspicuous set of composition factors in the statement of the proposition,

$$18_{4,2,3}, 8_{1,2,3}^2, 8_{1,2,4}, 6_{1,3}, 6_{4,1}, 2_1,$$

where y has 38 distinct eigenvalues on V_{\min} , and there exist elements of order 123 cubing to y and with 40 distinct eigenvalues, but none with 38. If ζ is a primitive root of unity then y can be chosen to have the following eigenvalues.

Module	Eigenvalues
2_1	$\zeta^{\pm 1}$
$6_{1,3}$	$\zeta^{\pm 1}, \zeta^{\pm 17}, \zeta^{\pm 19}$
$6_{4,1}$	$\zeta^{\pm 12}, \zeta^{\pm 14}, \zeta^{\pm 16}$
$8_{1,2,3}$	$\zeta^{\pm 5}, \zeta^{\pm 7}, \zeta^{\pm 11}, \zeta^{\pm 13}$
$8_{1,2,4}$	$\zeta^{\pm 10}, \zeta^{\pm 12}, \zeta^{\pm 16}, \zeta^{\pm 18}$
$18_{4,2,3}$	$\zeta^{\pm 2}, \zeta^{\pm 3}, \zeta^{\pm 4}, \zeta^{\pm 8}, \zeta^{\pm 9}, \zeta^{\pm 10}, \zeta^{\pm 14}, \zeta^{\pm 15}, \zeta^{\pm 20}$

There exists an element \hat{y} of order 123 cubing to y and stabilizing all eigenspaces except for the $\zeta^{\pm 1}$ -eigenspaces.

Since $8_{1,2,3}$ is the only composition factor to occur with multiplicity greater than 1, any other factor in the socle must be a summand. The module $8_{1,2,3}$ only has extensions with 2_1 , $6_{1,3}$ and $8_{1,2,4}$ from the composition factors of $V_{\min} \downarrow_H$, and so the structure of $V_{\min} \downarrow_H$ must be $W \oplus 6_{4,1} \oplus 18_{4,2,3}$, where W consists of the remaining factors.

Since \hat{y} stabilizes all but the $\zeta^{\pm 1}$ -eigenspaces, it fixes the $32 \oplus 6 \oplus 18$ decomposition above, and if \hat{Y} denotes an infinite subgroup of G containing \hat{y} and stabilizing the same subspaces of V_{\min} as \hat{y} (which exists since $123 > v(E_7) = 75$), then $X = \langle \hat{Y}, H \rangle$ certainly stabilizes the 6- and 18-dimensional summands of $V_{\min} \downarrow_H$. Thus H is contained inside an element of \mathcal{X}^σ . \square

12.2 Characteristic at least 5

We now let $p \geq 5$: since there are elements of orders $(p^a + 1)/2$ or $(p^a - 1)/2$, and one of these is odd, we assume that $p^a \leq 150$. As all unipotent classes are generic for V_{\min} for all $p \geq 29$, we only need consider

$$p^a = 5, 7, 11, 13, 17, 19, 23, 25, 49, 121, 125.$$

As for $\text{PSL}_2(p^a)$, there are some restrictions we can place on the possible actions of a unipotent element u , above and beyond appearing on [13, Table 7], given by Lemma 5.16. This yields twenty-nine possible non-generic classes for various primes, as given below.

- (i) $(A_3 + A_1)''$, $p = 5$, acting as $5^2, 4^8, 2^7$;
- (ii) $D_4(a_1) + A_1$, $p = 5$, acting as $5^6, 4, 3^4, 2^5$;
- (iii) $A_3 + A_2$, $p = 5$, acting as $5^6, 4^2, 3^4, 2^2, 1^2$;
- (iv) A_4 , $p = 5$, acting as $5^{10}, 1^6$;
- (v) $A_3 + A_2 + A_1$, $p = 5$, acting as $5^6, 4^4, 2^5$;
- (vi) $A_4 + A_1$, $p = 5$, acting as $5^{10}, 2^2, 1^2$;
- (vii) $A_4 + A_2$, $p = 5$, acting as $5^{10}, 3^2$;
- (viii) $(A_5)''$, $p = 7$, acting as $7^2, 6^7$;
- (ix) $D_4 + A_1$, $p = 7$, acting as $7^6, 2^5, 1^4$;
- (x) $D_5(a_1)$, $p = 7$, acting as $7^6, 3^2, 2^2, 1^4$;
- (xi) $(A_5)'$, $p = 7$, acting as $7^4, 6^4, 1^4$;

- (xii) $A_5 + A_1$, $p = 7$, acting as $7^4, 6^3, 5^2$;
- (xiii) $D_5(a_1) + A_1$, $p = 7$, acting as $7^6, 4, 2^5$;
- (xiv) $D_6(a_2)$, $p = 7$, acting as $7^6, 5^2, 4$;
- (xv) $E_6(a_3)$, $p = 7$, acting as $7^6, 5^2, 1^4$;
- (xvi) $E_7(a_5)$, $p = 7$, acting as $7^6, 6, 4^2$;
- (xvii) A_6 , $p = 7$, acting as 7^8 ;
- (xviii) $E_7(a_4)$, $p = 11$, acting as $11^2, 10, 8, 6, 4^2, 2$;
- (xix) D_6 , $p = 11$, acting as $11^4, 10, 1^2$;
- (xx) $E_6(a_1)$, $p = 11$, acting as $11^4, 5^2, 1^2$;
- (xxi) $E_7(a_3)$, $p = 11$, acting as $11^4, 10, 2$;
- (xxii) E_6 , $p = 13$, acting as $13^4, 1^4$;
- (xxiii) $E_7(a_3)$, $p = 13$, acting as $13^2, 12, 10, 6, 2$;
- (xxiv) $E_7(a_2)$, $p = 13$, acting as $13^4, 4$;
- (xxv) $E_7(a_2)$, $p = 17$, acting as $17^2, 10, 8, 4$;
- (xxvi) $E_7(a_1)$, $p = 17$, acting as $17^2, 16, 6$;
- (xxvii) $E_7(a_1)$, $p = 19$, acting as $19^2, 12, 6$;
- (xxviii) E_7 , $p = 19$, acting as $19^2, 18$;
- (xxix) E_7 , $p = 23$, acting as $23^2, 10$.

Thus we need to consider $p = 5, 7, 11, 13, 17, 19, 23$, and we will examine each in turn.

Proposition 12.4 Suppose that $p = 5$.

- (i) If $a = 1$ then H fixes a 2-space on V_{\min} .
- (ii) For $a = 2$, one of the following holds: H and $N_{\bar{G}}(H)$ fix a 2-space on V_{\min} ; H fixes a line on $L(G)$ or H stabilizes a 4-space on V_{\min} that has a positive-dimensional stabilizer; H stabilizes an \mathfrak{sl}_2 -subalgebra of $L(G)$.
- (iii) If $a = 3$ then an element of order 63 in H is a blueprint for V_{\min} , and hence H is a blueprint for V_{\min} .

Proof: Only the element of order 3 is important here, and it has trace one of $-25, -7, 2, 20$, with the last case not possible, and so we get

$$4, 2^{26}, \quad 4^7, 2^{14}, \quad 4^{10}, 2^8.$$

As $P(4) = 4/2/4$, each of these must fix a 2-space on V_{\min} , as claimed.

When $a = 2$, there are 106 conspicuous sets of composition factors for $V_{\min} \downarrow_H$, but fifty of these have no corresponding set of composition factors for $L(G)$, so can be ignored. None of the 106 sets is definable

over \mathbb{F}_5 , so if H stabilizes a line on $L(G)$ or a 2-space on V_{\min} then we are done. Removing those sets of factors with non-positive 2_i -pressure (and at least one 2_i) leaves eighteen, nine up to field automorphism of H . Exactly one of these has two possible sets of composition factors on $L(G)$, one of which has a single trivial and a single 8-dimensional, so that second option will be ignored as having pressure 0. Of the other eight, which all have a unique corresponding set of composition factors on $L(G)$, three have non-positive pressure and trivial factors, so that leaves us with six conspicuous sets of composition factors on V_{\min} . Up to field automorphism of H , these are

$$\begin{aligned} &12_{1,2}^2, 10_{1,2}, 4_2^2, 4_1^2, 2_1^3, \quad 12_{1,2}, 10_{1,2}, 6_{2,1}, 6_{1,2}^2, 4_2^3, 2_1^2, \quad 12_{1,2}, 10_{1,2}^2, 6_{2,1}, 6_{1,2}, 4_2, 4_1, 2_2, 2_1, \\ &12_{1,2}, 12_{2,1}, 10_{2,1}, 6_{2,1}, 6_{1,2}, 4_2, 4_1, 2_1, \quad 12_{1,2}^2, 10_{2,1}, 10_{1,2}, 6_{1,2}, 4_1, 2_1, \quad 12_{2,1}^2, 10_{1,2}, 6_{1,2}^3, 4_1. \end{aligned}$$

The simple modules with extensions with 2_1 are 4_2 , $6_{2,1}$ and $12_{1,2}$.

Case 1: This has pressure 1, and we may assume that we have a submodule of $P(12_{1,2})$ or $P(4_2)$ with three 2_1 s. The $\{2_1, 4_1, 4_2, 10_{1,2}, 12_{1,2}\}$ -radicals of these two modules are $10_{1,2}/12_{1,2}/2_1, 4_1, 10_{1,2}/12_{1,2}$ and $4_2/2_1/4_2$, so H fixes a 2-space on V_{\min} .

Case 2: The corresponding submodule of $P(4_2)$ in the second case is also $4_2/2_1/4_2$, so again this fixes a 2-space on V_{\min} .

Cases 3, 4: The 2_1 -pressure is 1, but the only simple module with multiplicity more than 1 is $10_{1,2}$, so 2_1 (and 2_2) split off, and the result holds again, and of course the same idea works for the fourth case.

Case 6: There are no extensions between the simple modules involved so $V_{\min} \downarrow_H$ is semisimple. The corresponding composition factors for $L(G) \downarrow_H$ have no extensions between them either and so the restriction is also semisimple, acting as

$$15_{1,2}^{\oplus 3} \oplus 15_{2,1}^{\oplus 2} \oplus 9^{\oplus 3} \oplus 5_1 \oplus 5_2 \oplus 3_1^{\oplus 4} \oplus 3_2^{\oplus 3}.$$

Write x for an element of order 13. Choosing ζ a primitive 13th root of unity appropriately (so that x acts on 2_1 with eigenvalues $\zeta^{\pm 1}$) the eigenvalues of x on V_{\min} are

$$1^4, (\zeta^{\pm 1})^7, (\zeta^{\pm 2})^6, (\zeta^{\pm 3})^3, (\zeta^{\pm 4})^6, \zeta^{\pm 5}, (\zeta^{\pm 6})^3.$$

Looking through the elements of order 26 in E_7 , we find one \hat{x} that squares to x , and if θ is a primitive 26th root of 1 with $\theta^2 = \zeta$, we have that the eigenvalues of \hat{x} are

$$1^4, (\theta^{\pm 1})^7, (\theta^{\pm 2})^6, (\theta^{\pm 3})^3, (\theta^{\pm 4})^6, \theta^{\pm 5}, (\theta^{\pm 6})^2, (-\theta^{\pm 6}).$$

This stabilizes the 4-space of V_{\min} stabilized by H (as well as the 6-spaces and the sum of the 12- and 10-spaces), so the stabilizer Y of this 4-space is either an almost simple group $\text{PGL}_2(25)$ modulo $Z(G)$ or H is not the socle of an almost simple maximal subgroup. The eigenvalues of \hat{x} on $L(G)$ are

$$1^{17}, (\theta^{\pm 1})^{10}, (\theta^{\pm 2})^{13}, (\theta^{\pm 3})^{12}, (\theta^{\pm 4})^6, (\theta^{\pm 5})^8, (-\theta^{\pm 5})^3, (\theta^{\pm 6})^4, (-\theta^{\pm 6})^2,$$

so $\langle H, \hat{x} \rangle$ is not $\text{PGL}_2(25)$ by Corollary 4.14. By Proposition 3.3, either H is contained in a member of \mathcal{X}^σ or H is contained in a copy of the Rudvalis group Ru (which is really $2 \cdot Ru$) acting on V_{\min} as $28 \oplus 28^*$ by the table in [23]. Since this is incompatible with the action of H , we get that H is contained inside a member of \mathcal{X}^σ , and this 4-space stabilizer must itself be positive dimensional.

Case 5: The $10_{2,1}$ must split off as it has no extensions with the $12_{1,2}$, but the rest of the composition factors of $V_{\min} \downarrow_H$ can lie above it, and there is a unique module

$$10_{2,1} \oplus 12_{1,2}/2_1, 4_1, 6_{1,2}, 10_{1,2}/12_{1,2},$$

with u acting as $5^{10}, 3^2$, class $A_4 + A_2$. The action of u on the direct sum of the composition factors of $V_{\min} \downarrow_H$ has block structure $5^8, 4^2, 2^4$, and examining [13, Table 7], we see that there are only two possible actions for u : $5^{10}, 3^2$ and $5^{10}, 2^2, 1^2$. Thus the 4_1 and $6_{1,2}$ cannot be summands (as they both have a 4 in the action of u), and we assume that the 2_1 is not a submodule, so if $V_{\min} \downarrow_H$ is not the module above then only the $10_{1,2}$ can be removed from the non-simple summand. However, the $\{2_1, 4_1, 6_{1,2}, 12_{1,2}\}$ -radical of $P(12_{1,2})$ is

$$12_{1,2}/2_1, 4_1, 6_{1,2}/12_{1,2},$$

but with a $6_{1,2}$ quotient, not allowed. Thus $V_{\min} \downarrow_H$ is as above, and in particular the symmetric square of this has $L(G) \downarrow_H$ as a summand (since $S^2(V_{\min}) = L(G) \oplus L(2\lambda_1)$).

The composition factors of $L(G) \downarrow_H$ are

$$16, 15_{1,2}^2, 15_{2,1}^2, 8_{1,2}^2, 8_{2,1}^3, 3_1^2, 3_2^3, 1^2,$$

and u must act on $L(G)$ as $5^{26}, 3$. There are only six isomorphism types of indecomposable module appearing as a summand of $S^2(V_{\min} \downarrow_H)$ whose composition factors appear on the list above, and these have structures

$$3_1, \quad 15_{2,1}, \quad 8_{2,1}/1, 3_2/8_{2,1}, \quad 15_{2,1}/8_{2,1}/1, 3_2/8_{2,1}/15_{2,1}, \quad 15_{1,2}/8_{1,2}/1, 3_1/8_{1,2}/15_{1,2}, \quad 3_2/8_{2,1}, 16/3_2,$$

with the second module appearing only once. There is only one way to assemble these summands into a module with the right unipotent action and composition factors, and this is

$$15_{2,1}/8_{2,1}/1, 3_2/8_{2,1}/15_{2,1} \oplus 15_{1,2}/8_{1,2}/1, 3_1/8_{1,2}/15_{1,2} \oplus 3_2/8_{2,1}, 16/3_2 \oplus 3_1.$$

In particular, this has 3_1 as a summand and so this is an \mathfrak{sl}_2 -subalgebra by Proposition 4.17, and also 3_2 as a submodule and subalgebra, but not necessarily a copy of \mathfrak{sl}_2 . This completes the proof for $a = 2$.

Finally, let $a = 3$. Using the traces of semisimple elements of order up to 31 there are 434 conspicuous sets of composition factors, 146 up to field automorphism. Checking the traces of elements of order 63, we find that eleven of these are not conspicuous for elements of order 63, and the remaining 135 all have preimages of order $5 \cdot 63 = 315$ that have the same number of eigenspaces on V_{\min} . Since $315 > v(E_7) = 75$ (for odd-order elements) we have that elements of order 63 in H are always blueprints for V_{\min} , and hence H is as well. \square

In the introduction we claimed that this potential $\mathrm{SL}_2(25)$ is maximal if it exists, so we must show that it is not contained in any positive-dimensional subgroup. By consideration of composition factors and summand dimensions, it can only lie inside a D_6 -parabolic; then one can proceed either by showing that the 12-dimensional factor cannot support a symmetric bilinear form, or by noting that if the subgroup lies inside the D_6 -parabolic then another lies inside the D_6 -Levi, hence acting semisimply, but the action of the unipotent element would be $5^8, 4^2, 2^4$, not appearing in [13, Table 7].

The next case is $p = 7$, where we cannot prove that there are no maximal $\mathrm{SL}_2(7)$ s in all cases.

Proposition 12.5 Suppose that $p = 7$.

- (i) Let $a = 1$. If H does not fix either a 2-space on V_{\min} , or a 1-space on $L(G)$, then the action of H on V_{\min} and $L(G)$ are

$$P(6)^{\oplus 2} \oplus P(4) \oplus 6 \oplus 4^{\oplus 2} \quad \text{and} \quad 7^{\oplus 5} \oplus P(5)^{\oplus 3} \oplus P(3)^{\oplus 3} \oplus 5 \oplus 3^{\oplus 3}$$

respectively.

- (ii) Let $a = 2$. If H is not a blueprint for V_{\min} then either $N_{\bar{G}}(H)$ fixes a 2-space on V_{\min} or a line on $L(G)$.

Proof: We start with $a = 1$. The conspicuous sets of composition factors are

$$4, 2^{26}, \quad 4^7, 2^{14}, \quad 6, 4^9, 2^7, \quad 6^3, 4^7, 2^5, \quad 6^4, 4^3, 2^{10}, \quad 6^5, 4^5, 2^3, \quad 6^6, 4, 2^8, \quad 6^7, 4^3, 2.$$

As the projective indecomposable modules are

$$P(2) = 2/4, 6/2, \quad P(4) = 4/2, 4/4, \quad P(6) = 6/2/6,$$

we look through the list above, checking to see whether we have enough 4s and 6s (three of the first or two of the second) to cover all 2s; this leaves the sixth and eighth cases of $6^5, 4^5, 2^3$ and $6^7, 4^3, 2$ to deal with.

Case 8: We switch to $L(G)$, and there is only one corresponding set of composition factors on $L(G)$, namely $7, 5^{15}, 3^{10}, 1^{21}$, which means that H fixes a line on $L(G)$.

Case 6: The only possible structure that does not fix a 2-space on V_{\min} and also yields a unipotent action from the list at the start of this section is $P(6)^{\oplus 2} \oplus P(4) \oplus 6 \oplus 4^{\oplus 2}$, with u lying in class $E_7(a_5)$. The factors of $L(G) \downarrow_H$ aren't uniquely determined, and can be any one of

$$7^5, 5^{15}, 3^2, 1^{17}, \quad 7^2, 5^{18}, 3^5, 1^{14}, \quad 7^8, 5^7, 3^{12}, 1^6, \quad 7^5, 5^{10}, 3^{15}, 1^3.$$

The first three of these must fix a line on $L(G)$, but the last one could in theory not, with module action

$$7^{\oplus 5} \oplus P(5)^{\oplus 3} \oplus P(3)^{\oplus 3} \oplus 5 \oplus 3^{\oplus 3},$$

this action compatible with the action of u on V_{\min} .

Now let $a = 2$, and recall that $L = \text{SL}_2(7)$. The eigenvalues of an element y of order 25 on V_{\min} are enough to determine the semisimple class of E_7 to which y belongs. This allows us to apply Lemma 4.12 to see that if there is an A_1 subgroup with 24-restricted composition factors, then any subgroup H of G whose composition factors on V_{\min} match the restriction of this A_1 to $\text{SL}_2(49)$ is a blueprint for V_{\min} .

There are 150 conspicuous sets of composition factors for $V_{\min} \downarrow_H$, and this is too many to analyse one at a time, although none of these is definable over \mathbb{F}_7 , so we can eliminate those of negative 2_i -pressure. Of the 150, only 92 of them have a corresponding set of composition factors for $L(G)$, and only 42 of these have either no trivial factors or positive pressure. Of these 42, only 26 have either no 2_i or positive 2_i -pressure for $i = 1, 2$, so we have thirteen conspicuous sets of composition factors left to deal with, up to field automorphism. Six of these have a 2_1 composition factor:

$$\begin{aligned} &6^5, 6_{1,2}, 4_2^4, 2_1^2, \quad 18_{1,2}^2, 10_{2,1}, 4_1, 4_2, 2_1, \quad 28_{1,2}, 10_{2,1}, 6_2^2, 4_1, 2_1, \\ &14_{1,2}, 10_{1,2}, 6_2^2, 6_{1,2}, 4_2^3, 2_1, \quad 18_{2,1}, 10_{2,1}^2, 6_1, 6_{1,2}, 4_2, 2_1, \quad 14_{2,1}, 10_{2,1}^3, 6_{1,2}, 4_2, 2_1. \end{aligned}$$

Case 1: Consider the diagonal A_1 inside the $A_1 A_1$ maximal subgroup of G , acting along each factor as $L(1)$: this acts on V_{\min} as

$$(L(6) \otimes L(3)) \oplus (L(2) \otimes L(5)) \oplus (L(4) \otimes L(1)),$$

which has factors $L(5)^5, L(3)^4, L(7)^2, L(9)$, up to field automorphism the same as the first case above.

Case 4: Inside $C_3 G_2$, consider an A_1 subgroup X acting along the first factor as $L(5)$ and along the second as $L(2) \oplus L(8)$. The action of X on V_{\min} is

$$L(5)/L(7)/L(5) \oplus L(3) \oplus L(11) \oplus L(13) \oplus L(3)/L(9)/L(3),$$

so the restriction to $\mathrm{SL}_2(49)$ has composition factors $14_{2,1}, 10_{2,1}, 6_1^2, 6_{2,1}, 4_1^3, 2_2$, up to field automorphism a match for the fourth case.

Case 5: For the fifth case, we note that the composition factors are the same as

$$2_2 \otimes ((6_1 \otimes 2_2) \oplus 5_1/4_{1,2}/5_1 \oplus) \oplus 4_2 :$$

inside $A_1 F_4$, let X be a copy of A_1 acting along the first factor as $L(7)$ and along the second factor as $L(12) \oplus L(4)/L(8)/L(4)$. This subgroup of F_4 exists inside the $A_1 C_3$ subgroup, acting irreducibly on the minimal modules of both subgroups as $L(7)$ and $L(5)$. The composition factors of $V_{\min} \downarrow_X$ match the fifth case.

Case 6: For the sixth case, as in the fifth, we note that the composition factors are the same as

$$2_2 \otimes (7_1 \oplus 5_1/4_{1,2}/5_1 \oplus 5_1) \oplus 4_2 :$$

inside $A_1 F_4$, let X be a copy of A_1 acting along the first factor as $L(7)$ and along the second factor as $L(6) \oplus L(4)/L(8)/L(4) \oplus L(4)$. This subgroup of F_4 exists inside the $A_1 G_2$ subgroup, acting irreducibly on the minimal modules of both subgroups as $L(1)$ and $L(6)$. The composition factors of $V_{\min} \downarrow_X$ match the sixth case.

Since these embeddings have factors up to $L(21)$, in all cases H is a blueprint for V_{\min} by Lemma 4.12.

Case 3: We now address the third case. Inside $C_3 G_2$, let X be an A_1 subgroup acting along the C_3 as $L(1) \oplus L(21)$ and acting along G_2 as $L(6)$. The action of X on V_{\min} is

$$L(27) \oplus L(5)/L(7)/L(5) \oplus L(3) \oplus L(29).$$

Up to field automorphism, the composition factors match the third case. While these are not 24-restricted, they are close: checking the weight spaces against the eigenvalues of the $\mathrm{SL}_2(49)$ contained within it, all weight spaces that have the same eigenvalues when restricted to $\mathrm{SL}_2(49)$ are contained within the $L(27) \oplus L(29)$. Thus if H is not a blueprint then H is contained in a positive-dimensional subgroup Y with composition factors of dimension 38, 6, 6, 4, 2. Notice that therefore Y , and hence H , must lie in either $C_3 G_2$ itself – and we know from above that in that case H is a blueprint for V_{\min} – or inside $A_1 F_4$, but it is easily seen to not be possible to place H inside this subgroup by the action of H on V_{\min} . Thus H is indeed a blueprint for V_{\min} .

Case 2: Here if H is not semisimple – and hence stabilizes a 2-space of V_{\min} – then the action of H on V_{\min} is

$$18_{1,2}/2_1, 4_1/18_{1,2} \oplus 10_{2,1} \oplus 4_2;$$

we claim that such a subgroup H does not lie in a positive-dimensional subgroup of G . To see this, firstly the dimensions of the composition factors are not compatible with coming from any maximal parabolic, so

that H must be contained in a reductive maximal subgroup, where the dimensions and multiplicities exclude A_7 and A_2 , and it is easy to see that it doesn't lie in the $A_1 A_1$.

For H to lie in $A_1 D_6$, $18_{1,2}$ would have to lie in the product of a module for SL_2 of dimension 2 and a module for PSL_2 of dimension 12: this is possible, but only with 2_1 being tensored by $2_1 \otimes 6_2$, and this yields 6_2 as well, not in $V_{\min} \downarrow_H$.

If H lies in $A_2 A_5$ then we must have that H acts on the natural modules along each factor as 3_1 and 6_2 , whence the module $(00, \lambda_3)$ is $4_2/6_{1,2}/4_2 \oplus 6_2$, obviously not correct.

If H lies in $G_2 C_3$, the the tensor product of the two minimal modules for these groups must be $18_{1,2}^2, 4_i$ for some i , and this is obviously impossible.

For H to lie in $A_1 G_2$, $18_{1,2}$ would have to be a composition factor of the tensor product of a module for $SL_2(49)$ of dimension 4 and a module for $PSL_2(49)$ of dimension 7, not possible.

We finally have H in $A_1 F_4$, where H must act on the natural module as 2_i for some i , yielding 4_i as a composition factor of $V_{\min} \downarrow_H$, and the rest of the module must be $2_i \otimes M$ for some 26-dimensional module M for $PSL_2(49)$, and this cannot yield $18_{1,2}^2$, so H cannot embed in this subgroup either.

Having proved this, we now show that the stabilizer of the 4_2 submodule is positive dimensional, a contradiction. There are eight elements of order 50 in a maximal torus of G squaring to y of order 25 and preserving the eigenspaces making up the 4_2 : these eight elements generate a subgroup $Z_{50} \times Z_2 \times Z_2$ of order 200, and so the stabilizer of the 4-space contains H as a subgroup of index at least 4, ruling out the possibility that it is almost simple with socle H . Thus H is contained in a member of \mathcal{X}^σ by Proposition 3.3, a contradiction and we are done.

We now give the seven conspicuous sets of composition factors with no 2_i in them.

$$\begin{aligned} &18_{2,1}^2, 6_1, 6_{2,1}, 4_1^2, \quad 18_{2,1}, 14_{2,1}, 10_{2,1}, 6_{2,1}, 4_1^2, \quad 42_{1,2}, 6_{2,1}, 4_1^2, \quad 28_{1,2}, 14_{1,2}, 6_1, 4_1^2 \\ &12_{2,1}^2, 10_{1,2}, 6_{1,2}^3, 4_1, \quad 28_{2,1}, 12_{2,1}, 6_{2,1}^2, 4_1, \quad 28_{1,2}, 18_{2,1}, 10_{1,2}. \end{aligned}$$

Cases 1 and 2: Inside $A_2 A_5$, let a subgroup X of type A_1 act along the two factors as $L(14)$ and $L(5)$ respectively. The composition factors of $V_{\min} \downarrow_X$ are $L(21)^2, L(5), L(3)^2, L(9)$. This is the first case.

Inside $C_3 G_2$, consider an A_1 subgroup X acting along the first factor as $L(5)$ and along the second as $L(14) \oplus L(8)$. The action of X on V_{\min} is

$$L(19) \oplus L(13) \oplus L(11) \oplus L(3)/L(9)/L(3).$$

The composition factors of X on V_{\min} match the second case.

Case 3: Inside $C_3 G_2$, consider an A_1 subgroup X acting along the first factor as $L(5)$ and along the second as $L(42)$. The action of X on V_{\min} is

$$L(47) \oplus L(3)/L(9)/L(3),$$

a match for the third case, but of course these are not 24-restricted, but do satisfy the second condition of Lemma 4.12, so that H is a blueprint for V_{\min}

Case 4: Inside $F_4 A_1$, let X denote an A_1 subgroup acting along the second factor as $L(1)$, and along the first factor as $L(4) \oplus L(44)$, which exists inside the $A_1 G_2$ subgroup of F_4 . The action of X on V_{\min} is

$$L(43) \oplus L(45) \oplus L(5) \oplus L(3)^{\oplus 2},$$

so this is the fourth case. Of course, these are not 24-restricted, so we proceed as in the second case of the previous set of composition factors, looking for elements of order 50 squaring to y and stabilizing the eigenspaces that comprise the 4_1 in the socle. Again, we find a subgroup $Z_{50} \times Z_2 \times Z_2$, and we conclude as before that the stabilizer of the 4_1 is a positive-dimensional subgroup of G .

We claim that H is a blueprint for V_{\min} . With the dimensions and multiplicities of the composition factors, the only maximal positive-dimensional subgroups it can lie in are D_6A_1 , C_3G_2 , A_1G_2 , A_1F_4 and A_1A_1 , with the last one clearly impossible.

If $H \leq D_6A_1$ then $14_{1,2} \oplus 6_1 \oplus 4_1$ is a tensor product of a 12-dimensional and a 2-dimensional module, so must be $2_1 \otimes (7_2 \oplus 5_1)$. Thus H lies inside the product of the A_1 and a product of two orthogonal groups, $\text{Spin}_7 \times \text{Spin}_5$, and there is a unique action of an A_1 subgroup inside these of acting as $L(42) \oplus L(0)$ and $L(42)$ on the two relevant modules of Spin_7 , and as $L(3)$ and $L(4)$ on the two modules of Spin_5 . This A_1 fixes the same subspaces of V_{\min} as H , so H is a blueprint for V_{\min} . The same statement holds from above for A_1F_4 .

If $H \leq C_3G_2$ then a similar analysis shows that H acts on the minimal modules of the two factors as $2_1 \oplus 4_1$ and 7_2 respectively, and the A_1 acting along each factor as $L(1) \oplus L(3)$ and $L(42)$ again stabilizes the same subspaces of V_{\min} as H , so H is a blueprint for V_{\min} .

We are left with A_1G_2 , where in order to find the $28_{1,2}$ we must have that H acts on the two natural modules as 2_1 and 7_2 respectively, so that the factor $3 \otimes 10$ in $V_{\min} \downarrow_{A_1G_2}$ yields $28_{1,2}$, but then the other factor of $1 \otimes 01$ yields two copies of $6_{1,2}$, not correct. (Indeed, this is how we get the sixth case above.)

Thus, whenever H is a subgroup of a positive-dimensional subgroup of G , it is a blueprint for V_{\min} , as needed.

Case 5: Inside the maximal subgroup A_1G_2 , let X be an A_1 subgroup acting along the first factor as $L(7)$ and along the second as $L(2)^{\oplus 2} \oplus L(0)$: the composition factors of X on V_{\min} are $L(9)^3, L(11), L(21), L(23)^2$, matching up with the fifth case, up to field automorphism. Since they are all 24-restricted, H is a blueprint for V_{\min} by Lemma 4.12.

Case 6: Inside the same subgroup A_1G_2 , let X instead be an A_1 subgroup acting along the first factor as $L(7)$ and along the second as $L(6)$: the composition factors of X on V_{\min} are $L(3), L(9)^2, L(17), L(27)$, matching up with the sixth case, but no longer 24-restricted, but H is still a blueprint for V_{\min} by Lemma 4.12(ii).

Case 7: Inside the maximal A_1A_1 subgroup, take a diagonal A_1 as we have done before, but this time acting as $L(1)$ and $L(7)$ along the two factors. The composition factors of this on V_{\min} are $L(27), L(13), L(37)$, of course not 24-restricted, and even Lemma 4.12(ii) doesn't work in this case. If H is contained in a positive-dimensional subgroup other than A_1A_1 then the dimensions of the composition factors and multiplicities show that it can only come from A_1G_2 , and in order to get $28_{1,2}$ appearing, H must act along A_1 as 2_1 and G_2 as 7_2 . However, the other factor of dimension 28 must have $6_{1,2}$ as a composition factor, not allowed. Thus $H \leq A_1A_1$, and so H is a blueprint for V_{\min} .

It remains to show that H is always contained inside a member of \mathcal{X} . Of course, since $V_{\min} \downarrow_H$ is multiplicity free, it is semisimple, and so the $10_{1,2}$ is a submodule. As with previous cases, we find more than one element of order 50 squaring to y and stabilizing the eigenspaces that comprise $10_{1,2}$. The subgroup generated by these is $Z_{50} \times Z_2$, and we wish to apply Corollary 4.14, so we need to find an element \hat{y} of order 50 in this subgroup whose action on $L(G)$ has no (-1) -eigenspace, but this is easy: its eigenspaces are

$$\begin{aligned} &1^7, (\theta^{\pm 2})^6, (\theta^{\pm 4})^4, (\theta^{\pm 5})^3, (\theta^{\pm 6})^2, (\theta^{\pm 7})^3, (\theta^{\pm 8})^2, (\theta^{\pm 9})^5, (\theta^{\pm 11})^6, (\theta^{\pm 13})^6, \\ &(\theta^{\pm 15})^5, (\theta^{\pm 16})^4, (\theta^{\pm 17})^3, (\theta^{\pm 18})^2, (\theta^{\pm 19})^2, (\theta^{\pm 20})^4, (\theta^{\pm 21})^5, (\theta^{\pm 22})^5, (\theta^{\pm 24})^6, \end{aligned}$$

where θ is a primitive 50th root of unity and $y = \hat{y}^2$ acts on 2_1 with eigenvalues $\theta^{\pm 2}$.

We have thus shown that all cases are blueprints for V_{\min} , fix a 2-space on V_{\min} , or fix a line on $L(G)$, as needed. \square

Proposition 12.6 Suppose that $p = 11$.

- (i) Let $a = 1$. If H is not a blueprint for V_{\min} , then $N_{\bar{G}}(H)$ fixes a 2-space on V_{\min} or a line on $L(G)$.
- (ii) Let $a = 2$. If H is not a blueprint for V_{\min} , then $N_{\bar{G}}(H)$ fixes a 2-space on V_{\min} .

Proof: Let $a = 1$. As $p = 11$ the action of u is one of cases (xviii) to (xxi) in the list above. In the first unipotent class, $E_7(a_4)$ acting as $11^2, 10, 8, 6, 4^2, 2$, there are single blocks of size 10, 8, 6 and 2, which must come from simple summands of those dimensions. Since we cannot have a faithful indecomposable module of dimension $11 + 4 = 15$, the 4s must also come from simple summands, and so $V_{\min} \downarrow_H$ is a single projective plus a semisimple module. The conspicuous such sets of composition factors yield

$$P(10) \oplus 10 \oplus 8 \oplus 6 \oplus 4^{\oplus 2} \oplus 2 \quad \text{and} \quad P(4) \oplus 10 \oplus 8 \oplus 6 \oplus 4^{\oplus 2} \oplus 2.$$

The second of these has corresponding set of factors on $L(G)$ given by $9^8, 7^2, 5^7, 1^{12}$, and the action of u on $L(G)$ is $11^8, 9, 7^2, 5^2, 3^4$. Blocks of size 3 come from, up to duality,

$$3, \quad 7, 9/1, 3, 5, \quad 3, 5, 7/5, 7, 9,$$

and so H cannot embed with these factors and this action of u . (The other set of composition factors yields

$$11^{\oplus 4} \oplus P(9) \oplus P(7) \oplus 9 \oplus 7^{\oplus 2} \oplus 5^{\oplus 2} \oplus 3^{\oplus 4}.)$$

If u comes from class D_6 , acting as $11^4, 10, 1^2$, then the 10 must come from a simple summand, and the 1^2 comes from (up to duality) $6/6, 4/8$ or $2/10$. The sum of one of these plus its dual, a single projective indecomposable module, and the 10, must be conspicuous. There are five such conspicuous sets of composition factors, all of which have corresponding sets of factors on $L(G)$, three of them having two different sets. The action of u on $L(G)$ is $11^{10}, 10^2, 1^3$, so certainly H has a trivial summand on $L(G)$, with the 10^2 coming from $(5/5)^{\oplus 2}, 3/7 \oplus 7/3$ or $1/9 \oplus 9/1$, the other 1^2 being either semisimple, $9/3 \oplus 3/9$ or $5/7 \oplus 7/5$, with the rest of the module being projective. There are 937 such sets of composition factors, and when taking the intersection of that list with those of the corresponding sets of composition factors to our list for $V_{\min} \downarrow_H$, we find two members:

$$P(4) \oplus 10/2 \oplus 2/10 \oplus 10 \quad \text{and} \quad P(6) \oplus (6/6)^{\oplus 2} \oplus 10,$$

with corresponding embeddings

$$11^{\oplus 4} \oplus P(5) \oplus P(3) \oplus 3/9 \oplus 9/3 \oplus 3/7 \oplus 7/3 \oplus 1 \quad \text{and} \quad 11^{\oplus 6} \oplus P(5) \oplus P(3) \oplus 3/7 \oplus 7/3 \oplus 1^{\oplus 3}.$$

Of course, these fix a line on $L(G)$, as needed.

If u comes from class $E_6(a_1)$ acting as $11^4, 5^2, 1^2$, then the two blocks of size 5 must come from summands of dimension 16, so $4, 6/6 \oplus 6/4, 6$, and the two 1s must come from summands of dimension 12. The conspicuous such sets of composition factors yield

$$4, 6/6 \oplus 6/4, 6 \oplus 10/2 \oplus 2/10, \quad 4, 6/6 \oplus 6/4, 6 \oplus 4/8 \oplus 8/4, \quad 4, 6/6 \oplus 6/4, 6 \oplus (6/6)^{\oplus 2}.$$

Of these, only the last has a corresponding set of composition factors on $L(G)$, and this is $11^2, 9^8, 5^5, 1^{14}$; however, u acts as $11^{10}, 9, 5^2, 3, 1$ on $L(G)$, so we need a 3 as a summand of $L(G) \downarrow_H$, not possible. Thus H does not embed with u from this class.

Finally, if u acts as $11^4, 10, 2$, coming from $E_7(a_3)$, then the 2 and 10 must come from summands of the same dimension, so we need two projectives plus $10 \oplus 2$. There are three such conspicuous sets of composition factors, yielding

$$P(4)^{\oplus 2} \oplus 10 \oplus 2, \quad P(4) \oplus P(10) \oplus 10 \oplus 2, \quad P(6) \oplus P(10) \oplus 10 \oplus 2.$$

Each of these has corresponding sets of composition factors on $L(G)$, with the third having two. However, u acts on $L(G)$ with blocks $11^{11}, 9, 3$, so $L(G) \downarrow_H$ is the sum of $9 \oplus 3$ and projectives, and since we have 11^{11} in the action of u , the number of summands of $L(G) \downarrow_H$ that are either 11s or $P(1)$ s must be odd. In particular, any trivial composition factors lie either in $P(1)$ s or $P(9)$ s. The four corresponding sets of factors are

$$9^8, 7^3, 5^7, 1^{12}, \quad 11^4, 9^3, 7^4, 5^3, 3^6, 1, \quad 11^3, 9^2, 7^6, 5^5, 3^5, \quad 11^2, 9^4, 7^7, 5, 3^6, 1^3;$$

the first and second cases need an odd number of $P(1)$ s, and the second case can have no $P(9)$ s as it has no 3s, leading to a contradiction. The fourth case cannot work with the unipotent class either, but the third case yields

$$11^{\oplus 3} \oplus P(7)^{\oplus 2} \oplus P(5) \oplus P(3) \oplus 9 \oplus 3.$$

Thus $P(6) \oplus P(10) \oplus 10 \oplus 2$ is the only acceptable embedding of H into V_{\min} with this unipotent action.

Now let $a = 2$, and recall that L is a copy of $\mathrm{SL}_2(11)$ inside H . We have traces of semisimple elements of order up to 40, and can use the preimage trick from Section 4.2 to find traces of elements of orders 60 and 120 as well. We would like that the eigenvalues of an element y of order 120 on V_{\min} uniquely determine the semisimple class of E_7 to which y belongs. This is not true in general for all classes, but will be true for the particular classes that arise from conspicuous sets of composition factors for $V_{\min} \downarrow_H$, by a check using the preimage trick.

Suppose that u comes from class $E_7(a_4)$, so that the composition factors of $V_{\min} \downarrow_L$ are $10^3, 8, 6, 4^2, 2$. There are, up to field automorphism, twenty conspicuous sets of composition factors for $V_{\min} \downarrow_H$, using semisimple elements of order up to 40; using the preimage trick, we can eliminate seven of these from contention, as they fail either the trace of an element of order 60 or 120, leaving thirteen. Note that, for these thirteen remaining sets of composition factors, the eigenvalues of an element y of order 120 on V_{\min} determine the semisimple class to which y belongs.

Nine of the thirteen have a 2-dimensional composition factor, and are

$$14_{1,2}, 10_2^3, 4_2^2, 2_1^2, \quad 10_1^3, 8_1, 6_1, 4_1^2, 2_1, 2_2, \quad 22_{2,1}, 18_{2,1}^{(1)}, 6_1, 4_1^2, 2_1, \quad 22_{2,1}, 14_{2,1}, 10_1, 4_1^2, 2_1, \quad 10_1^3, 8_1, 6_1, 6_{2,1}, 4_1, 2_1,$$

$$30_{1,2}^{(1)}, 10_2, 10_{1,2}, 4_1, 2_1, \quad 22_{1,2}, 18_{1,2}^{(1)}, 10_{1,2}, 4_1, 2_1, \quad 18_{1,2}^{(1)}, 10_2^2, 6_2, 6_{1,2}, 4_2, 2_1, \quad 22_{1,2}, 18_{1,2}^{(2)}, 10_2, 4_2, 2_1.$$

(Recall that $18_{i,j}^{(1)} = 2_i \otimes 9_j$, $18_{i,j}^{(2)} = 3_i \otimes 6_j$ and $30_{i,j}^{(1)} = 3_i \otimes 10_j$.) The simple modules with non-trivial extensions with 2_1 are 10_2 , $18_{2,1}^{(1)}$ and $30_{1,2}^{(1)}$, and in order for H not to stabilize a 2-space, one of these must occur with multiplicity 2: thus all cases must fix a 2-space on V_{\min} except for the first and eighth. However, even in the first case the $\{2_1, 4_2, 10_2, 14_{1,2}\}$ -radical of $P(10_2)$ is simply $10_2/2_1/10_2$, so there must be a 2_1 submodule of $V_{\min} \downarrow_H$, so that case also fixes a 2-space.

For the eighth case, up to field automorphism we find this inside D_6A_1 , by taking an A_1 subgroup acting on the natural modules for the two factors as $9_1 \oplus 3_1 = L(8) \oplus L(2)$ and $2_2 = L(11)$ respectively.

This yields $18_{2,1}^{(1)} \oplus 6_{2,1} = L(19) \oplus L(13)$, and for the spin module for the D_6 term, we need a module with unipotent action $11^2, 6, 4$ (one sees this from the entry for $D_6(a_1)$ in [13, Table 7]) and composition factors $10_1^2, 6_1, 4_1, 2_2 = L(9)^2, L(5), L(3), L(11)$ (obtained from the traces of semisimple elements), and so the restriction of the spin module to this A_1 subgroup is

$$L(9)/L(11)/L(9) \oplus L(5) \oplus L(3),$$

so we apply Lemma 4.12 to see that H is a blueprint for V_{\min} , as needed.

The remaining four conspicuous sets of composition factors, which have no 2-dimensional composition factor, are

$$36_{2,1}, 10_1, 6_{1,2}, 4_1, \quad 42_{1,2}, 10_1, 4_1, \quad 22_{2,1}, 10_1, 10_{2,1}, 8_1, 6_{1,2}, \quad 28_{2,1}, 22_{2,1}, 6_{2,1}.$$

For the last of these, consider a diagonal A_1 inside A_1G_2 , acting as $2_2 = L(11)$ along A_1 and as $7_1 = L(6)$ along the G_2 factor. The composition factors on V_{\min} are $L(39), L(21), L(13)$, matching up with the fourth case above, so we satisfy the conditions of Lemma 4.12. Consider an A_1 subgroup of G_2C_3 , acting as $7_2 = L(66)$ along the G_2 factor and as $6_1 = L(5)$ along the C_3 factor: the composition factors on V_{\min} are $L(71), L(9), L(3)$, matching up with the second case above. Again we apply Lemma 4.12, this time the second statement, and therefore H is a blueprint for V_{\min} , as claimed.

For the third case, we find an A_1 subgroup Y inside D_6A_1 that works: consider Y acting as $2_1 = L(1)$ along the A_1 factor, and as $9_1 \oplus 3_2 = L(8) \oplus L(22)$ along the second factor. This second A_1 is contained diagonally as an irreducible subgroup inside the product of orthogonal groups $O_9 \times O_3$, i.e., B_4A_1 , so its action on the 32-dimensional half-spin module is as the tensor product of the spins. The action on Spin_3 must be as $2_2 = L(11)$, and the action on the Spin_9 has unipotent factors $11, 5$ and can be seen to be $11_1 \oplus 5_1 = L(10) \oplus L(4)$. Thus the subgroup Y has composition factors

$$L(21), L(15), L(9), L(7), L(23),$$

and hence satisfies Lemma 4.12, with the $\text{SL}_2(121)$ inside Y having the same factors on V_{\min} as the third case.

We are left with $36_{2,1}, 10_1, 6_{1,2}, 4_1$. We firstly note that if there is such a subgroup H then $N_{\bar{G}}(H)$ is not contained in a positive-dimensional subgroup of \bar{G} : to see this, notice that since $V_{\min} \downarrow_H$ is multiplicity-free and contains a 36-dimensional composition factor, the only positive-dimensional subgroups that could contain it are G_2C_3 and A_1F_4 . If $H \leq G_2C_3$, the module $10 \otimes 100$ has dimension 42, so must restrict to $36_{2,1} \oplus 6_{1,2}$, but $36_{2,1} = 9_1 \otimes 4_2$ is not a composition factor of any tensor product of a 6-dimensional module and a 7-dimensional module, a contradiction. Since A_1F_4 acts on V_{\min} with factors $1 \otimes 0001$ and $3 \otimes 0000$, we see that if $H \leq A_1F_4$ then the projection of H along A_1 must act on the natural module as 2_1 , so that $L(3)$ restricts to H as 4_1 . However, again $36_{2,1}$ is not a composition factor of any tensor product of 2_1 and a module of dimension at most 26, so H cannot lie in A_1F_4 either. Thus H , and by extension $N_{\bar{G}}(H)$, cannot lie in a positive-dimensional subgroup of G .

We therefore must have that $N_{\bar{G}}(H)$ contains $\text{SL}_2(121)$ with index at most 2 by Proposition 3.3. Recalling that y is an element of order 120, chosen so that the eigenvalues of y on 2_1 are $\zeta^{\pm 1}$ for ζ a primitive 120th root of unity, we consider the elements of order 120 in a maximal torus of G squaring to y^2 , noting that the ζ^2 - and ζ^6 -eigenspaces of y^2 on V_{\min} are both 3-dimensional and coincide with the ζ - and ζ^3 -eigenspaces of y respectively. We thus look for elements that square to y^2 and preserve the ζ^2 - and ζ^6 -eigenspaces of y^2 ; of course, y is one of these elements, and we find four elements with 3-dimensional ζ - and ζ^3 -eigenspaces (and therefore four with 3-dimensional $(-\zeta)$ - and $(-\zeta^3)$ -eigenspaces), which together generate a subgroup

$Z_{120} \times Z_2 \times Z_2$ of the torus. Thus the stabilizer in G of the 4-dimensional submodule 4_1 of V_{\min} contains H with index at least 4, a contradiction, and so H doesn't exist by the above proof. This completes the proof of the proposition when u comes from class $E_7(a_4)$.

Suppose that u comes from class D_6 , and that the composition factors of $V_{\min} \downarrow_L$ are $10, 6^7, 4$: the trace of an element of order 5 is 6, and this is enough to seriously restrict the possibilities. Using other semisimple elements of order up to 20, we find up to field automorphism a single conspicuous set of composition factors, namely $10_1, 6_1^7, 4_1$, and this must be semisimple as there are no non-trivial extensions between the factors. However, this is incompatible with the action of u , so H cannot embed with this restriction to L . The other set of composition factors are the same as for $E_7(a_4)$, which we have already considered above.

Suppose that u comes from $E_7(a_3)$, so that $V_{\min} \downarrow_L$ has composition factors $10^3, 6^3, 4, 2^2$. Checking traces of elements of order up to 40 yields, up to field automorphism, only five conspicuous sets of composition factors, which are

$$10_1^3, 6_1^3, 4_1, 2_1^2, \quad 10_2^3, 6_2^3, 4_2, 2_1^2, \quad 10_1^3, 10_{2,1}, 6_1^2, 2_1, 2_2, \quad 10_1, 10_2^2, 6_1, 6_2^2, 6_{1,2}, 2_1, \quad 22_{1,2}, 10_2, 10_{1,2}, 6_2^2, 2_1.$$

The last two of these fail the traces of elements of order 60, so do not exist. The 2_1 -pressures of the remaining three are $-2, 1$ and -1 respectively, as only 10_2 from these simple modules has an extension with 2_1 , so only the second need not fix a 2-space. In this case the $\{2_1, 4_2, 6_2, 10_2\}$ -radical of $P(10_2)$ is $10_2/2_1/10_2$, so $V_{\min} \downarrow_H$ has a 2-dimensional submodule. \square

We now have that $p \geq 13$, for which we only need consider $a = 1$. For $p = 17, 19, 23$, if H is not in a member of \mathcal{X}^σ , we will prove that H fixes an \mathfrak{sl}_2 -subalgebra, and for $p = 19$ we will also get a Serre embedding (see Definition 4.7).

We begin with $p = 13$.

Proposition 12.7 Suppose that $p = 13$ and $a = 1$. If H is not a blueprint for V_{\min} then $N_G(H)$ either fixes a 2-space or 4-space, and in both cases lies in a member of \mathcal{X}^σ .

Proof: There are three possibilities for the action of u on V_{\min} , namely cases (xxii), (xxiii) and (xxiv) from the list above, with the last of these being semiregular.

In the first case, u acts as $13^4, 1^4$, and so $V_{\min} \downarrow_H$ is the sum of four modules of dimension 14, which are

$$2/12, \quad 4/10, \quad 6/8, \quad 8/6, \quad 10/4, \quad 12/2.$$

There are only two conspicuous sets of composition factors consisting of dual pairs of these, and they are

$$2/12 \oplus 12/2 \oplus 4/10 \oplus 10/4 \quad \text{and} \quad 4/10 \oplus 10/4 \oplus 6/8 \oplus 8/6.$$

Neither of these has a corresponding set of composition factors on $L(G)$, so H does not embed in G with u coming from class E_6 .

If u comes from class $E_7(a_3)$ then it acts as $13^2, 12, 10, 6, 2$. The single block of size 2 must come from a self-dual indecomposable module of dimension congruent to 2 modulo 13, and the two of these are 2 itself – so H fixes a 2-space on V_{\min} – or a 28-dimensional module $6, 8/6, 8$; from here, the blocks of sizes 6 and 10 must come from simple summands and the 12 comes either from a 12 or a $6/6$, yielding two possible sets of composition factors, neither of which is conspicuous, having trace -1 for an element of order 3. This completes the proof for the second action of u .

The final unipotent class to consider is the semiregular $E_7(a_2)$, acting as $13^4, 4$ on the minimal module. The single block of size 4 comes either from a summand 4 or from the indecomposable module $4, 6, 8, 10/4, 6, 8, 10$, which is conspicuous, but we saw above that it has no corresponding set of factors for $L(G)$. Hence $V_{\min} \downarrow_H$ has a 4 as a summand, with two projective indecomposable summands. The conspicuous such sets of composition factors yield

$$P(2) \oplus P(4) \oplus 4, \quad P(12) \oplus P(10) \oplus 4, \quad P(10) \oplus P(8) \oplus 4, \quad P(6) \oplus P(4) \oplus 4.$$

The first and second of these cannot occur because they do not have corresponding factors on $L(G)$.

In the fourth case we again switch to $L(G)$, and find two corresponding sets of composition factors, namely

$$13, 11, 9^2, 7^9, 5^3, 3^4, 1, \quad 11^3, 9^3, 7^5, 5^4, 3^6.$$

The action of u on $L(G)$ must be $13^{10}, 3$, and the single 3 in this action comes from a summand isomorphic to either 3 or $5, 7, 9/5, 7, 9$. The first case cannot occur as the single 1 must lie in a $P(11)$, but this cannot occur. In the second case, no trivials means there can be no $P(11)$ s, so we must have $P(3)^{\oplus 3}$ in $L(G) \downarrow_H$, but then there are no 3s or 9s remaining, so the summand contributing the 3 to the action of u cannot occur, a contradiction. Thus H cannot embed with these factors either.

In the third case, there are again two corresponding sets of composition factors on $L(G)$, namely

$$13, 11^3, 9^2, 7^7, 5, 3^4, 1^3, \quad 11^5, 9^3, 7^3, 5^2, 3^6, 1^2.$$

In the first of these, the single 5 means one has no $P(5)$ and at most one $P(7)$, but these are the only two projectives containing 7, so we cannot use up the seven 7s. The second case does have a unique possibility, however, of

$$P(11)^{\oplus 2} \oplus P(9) \oplus P(7) \oplus P(3) \oplus 3.$$

Although it has 3 as a summand, the presence of a $P(3)$ means that we cannot guarantee that it is an \mathfrak{sl}_2 -subalgebra of $L(G)$, although the $3 \oplus 3$ in the socle of $L(G) \downarrow_H$ does form a subalgebra.

Let x denote an element of order 14 in H and let ζ be a primitive 14th root of unity, arranged so that the eigenvalues of x on 4 are $\zeta^{\pm 1}, \zeta^{\pm 3}$. The eigenvalues of x on V_{\min} are

$$(-1)^8, (\zeta^{\pm 1})^9, (\zeta^{\pm 3})^8, (\zeta^{\pm 5})^7.$$

Let θ denote a primitive 28th root of unity with $\theta^2 = \zeta$, and we find $\hat{x} \in G$ such that $\hat{x}^2 = x$ and \hat{x} has eigenvalues

$$(\pm i)^4, (\theta^{\pm 1})^9, (\theta^{\pm 3})^8, (\theta^{\pm 5})^5, (-\theta^{\pm 5})^2$$

on V_{\min} . This stabilizes the eigenspaces intersecting the 4, and so \hat{x} stabilizes the 4-space stabilized by H .

The action of \hat{x} on $L(G)$ has a 21-dimensional 1-eigenspace but -1 is not an eigenvalue of \hat{x} : if $\bar{H} = \langle H, \hat{x} \rangle / Z(G) \cong \text{PSL}_2(13)$ then all of the composition factors of $L(G) \downarrow_{\bar{H}}$ must be the plus-type factors, in the notation of Section 4.4. Thus we extend the action of $H/Z(G) \cong \text{PSL}_2(13)$ on $L(G)$ uniquely to an action of $\text{PGL}_2(13)$, and doing so yields a trace of 11 on $L(G)$ for an element of order 4 in $\text{PGL}_2(13) \setminus \text{PSL}_2(13)$. However, 11 is not the trace of an element of order either 4 or 8 (in case it powers to the central involution) in G on $L(G)$, a contradiction, so that $\langle H, \bar{x} \rangle$, which stabilizes the 4-space, is not almost simple modulo $Z(G)$. Thus $N_{\bar{G}}(H)$ is contained inside a member of \mathcal{X}^σ by Proposition 3.3, as needed. \square

Proposition 12.8 Suppose that $p = 17$ and $a = 1$. If H is not a blueprint for V_{\min} then either $N_{\bar{G}}(H)$ stabilizes a 4-space on V_{\min} whose stabilizer not $N_{\bar{G}}(H)$, and hence $N_{\bar{G}}(H)$ is contained in a member of \mathcal{X}^σ , or H stabilizes an \mathfrak{sl}_2 -subalgebra of $L(G)$.

Proof: There are two non-generic unipotent classes, cases (xxv) and (xxvi) above, where there are single Jordan blocks of sizes 4, 6, 8, 10, 16. Apart from the simple modules of dimension congruent to 4, 6, 8, 10, 16 modulo 17, the self-dual indecomposable modules congruent to those dimensions have dimensions 72, 108, 144, 114 and 16 respectively, so only the 16 might not come from a simple summand.

For u belonging to class $E_7(a_2)$, so acting as $17^2, 10, 8, 4$, we therefore have a single projective plus $10 \oplus 8 \oplus 4$. Applying the traces of semisimple elements yields two possibilities:

$$P(16) \oplus 10 \oplus 8 \oplus 4 \quad \text{and} \quad P(10) \oplus 10 \oplus 8 \oplus 4.$$

The second of these does not yield an action on $L(G)$ as the traces do not match up, but the first of these has a unique set of composition factors on $L(G)$ which yields

$$17 \oplus P(15) \oplus P(11) \oplus 15 \oplus 11 \oplus 9 \oplus 7 \oplus 3^{\oplus 2}.$$

The subspace $3^{\oplus 2}$ is an H -invariant Lie subalgebra of $L(G)$, but we proceed as in the case of $p = 13$, finding an element of order 36 that preserves the 4-dimensional submodule.

Let x be an element of H of order 18 and ζ be a primitive 18th root of unity, arranging our choices so that the eigenvalues of x on 4 are $\zeta^{\pm 1}$ and $\zeta^{\pm 3}$. The eigenvalues of x on V_{\min} are

$$(-1)^6, (\zeta^{\pm 1})^6, (\zeta^{\pm 3})^7, (\zeta^{\pm 5})^6, (\zeta^{\pm 7})^6.$$

Letting θ denote a primitive 36th root of unity squaring to ζ , we find an element \hat{x} of order 36 in G with $\hat{x}^2 = x$ and with eigenvalues on V_{\min} given by

$$(\pm i)^3, (\theta^{\pm 1})^6, (\theta^{\pm 3})^7, (\theta^{\pm 5})^5, -\theta^{\pm 5}, (\theta^{\pm 7})^4, (-\theta^{\pm 7})^2.$$

We see immediately that \hat{x} preserves the 4-space stabilized by H , and we proceed as in Proposition 12.7, noting that -1 is not an eigenvalue of \hat{x} on $L(G)$, and so there is a unique extension of $\text{PSL}_2(17)$ to $\text{PGL}_2(17)$. However, an element of order 6 in $\text{PGL}_2(17) \setminus \text{PSL}_2(17)$ has trace 8 on $L(G)$. While 8 is the trace of an element of order 6 in G (but not an element of order 12 powering to the central involution), the square of this element needs trace 34 on $L(G)$, which is not a trace of an element of order 3 in H , a contradiction. We then proceed as in Proposition 12.7.

For u belonging to class $E_7(a_1)$, so acting as $17^2, 16, 6$, the 6 must come from a simple summand, but the 16 comes from either a simple summand or $8/8$. Thus our embedding of H is either a single projective plus $8/8 \oplus 6$ or a single projective plus $16 \oplus 6$. Using traces, the two options are

$$P(12) \oplus 16 \oplus 6 \quad \text{and} \quad P(4) \oplus 16 \oplus 6.$$

The second of these has no corresponding set of composition factors on $L(G)$, but the first has a unique set, which implies that $L(G) \downarrow_H$ is

$$17 \oplus P(15) \oplus P(11) \oplus P(7) \oplus 11 \oplus 3,$$

and so the 3 is an \mathfrak{sl}_2 -subalgebra by Proposition 4.17 and we complete the proof. \square

Proposition 12.9 Suppose that $p = 19$ and $a = 1$. If H is not a blueprint for V_{\min} then H stabilizes an \mathfrak{sl}_2 -subalgebra of $L(G)$, or H is a Serre embedding.

Proof: When $p = 19$, there are two non-generic unipotent classes, $E_7(a_1)$ and E_7 , cases (xxvii) and (xxviii) above. As $p \equiv 3 \pmod{4}$ there is a unique self-dual indecomposable module congruent to any given integer modulo p . For $E_7(a_1)$ the 12 and 6 in the action of u must therefore come from simple summands, leaving a single projective module of dimension 38. Only two possibilities yield conspicuous sets of composition factors, namely $P(16) \oplus 12 \oplus 6$ and $P(4) \oplus 12 \oplus 6$. The second of these has no corresponding set of composition factors on $L(G)$, with the first of these yielding the unique action

$$P(15) \oplus P(11) \oplus 19 \oplus 17 \oplus 11 \oplus 7 \oplus 3,$$

with the 3 being an \mathfrak{sl}_2 -subalgebra of $L(G)$ by Proposition 4.17.

The remaining case is u coming from the regular class, where as with the $E_7(a_1)$ case the 10 from the action of u must yield a simple summand, with the rest projective. There are again two conspicuous sets of composition factors for $V_{\min} \downarrow_H$, coming from $P(4) \oplus 18$ and $P(10) \oplus 18$. The first has no corresponding set of composition factors on $L(G)$, and the second has a single set, which since u is projective on $L(G)$, must be arranged so that $L(G) \downarrow_H$ is

$$19 \oplus P(15) \oplus P(11) \oplus P(3).$$

While the 3-dimensional submodule is a subalgebra of $L(G)$, it is not obviously an \mathfrak{sl}_2 -subalgebra because we cannot apply Proposition 4.17. This is a Serre embedding as defined in Definition 4.7, as needed. \square

The last case is $p = 23$ and the regular unipotent class, to conclude this section.

Proposition 12.10 Suppose that $p = 23$ and $a = 1$. If H is not a blueprint for V_{\min} then H stabilizes an \mathfrak{sl}_2 -subalgebra of $L(G)$.

Proof: The only non-generic class for $p = 23$ and V_{\min} is the regular class, with blocks $23^2, 10$, case (xxix) above. The 10 in the action of u must come from a simple summand, leaving a single projective module of dimension 46. Only two possibilities yield conspicuous sets of composition factors, namely $20, 18, 10, 4^2$ and $18^2, 10, 6, 4$. The first of these has no corresponding set of composition factors on $L(G)$, with the second of these yielding the unique action

$$P(19) \oplus P(11) \oplus 23 \oplus 15 \oplus 3,$$

with the 3 being an \mathfrak{sl}_2 -subalgebra of $L(G)$ by Proposition 4.17. \square

A Actions of maximal positive-dimensional subgroups on minimal and adjoint modules

In this appendix we collate information on the actions of the reductive and parabolic maximal subgroups of positive dimension on the minimal and adjoint modules for the algebraic groups F_4 , E_6 and E_7 that we have used in the text, other than those in Lemmas 2.4 and 2.5. These have been documented in many places, but we give them here as well for ease of reference.

We need information for F_4 and E_6 in characteristic 3, and for E_6 in characteristics 7 and 11. We list the composition factors of every maximal closed, connected subgroup of positive dimension in these characteristics, taken from [20], on V_{\min} , and $L(G)$ for $G = F_4$. We list the reductive subgroups first, and then the parabolics. Write M^\pm to mean a module M and its dual M^* .

We begin with the table for F_4 in characteristic 3.

Subgroup of F_4 for $p = 3$	Factors on V_{\min}	Factors on $L(G)$
B_4	1000, 0001	0100, 0001
$\tilde{A}_1 C_3$	(1, 100), (0, 010)	(2, 000), (0, 200), (1, 001)
$A_2 \tilde{A}_2$	(10, 10), (01, 01), (00, 11)	(11, 00), (00, 11), (10, 02), (01, 20), (00, 00) ²
$A_1 G_2$	(2, 10), (4, 00)	(2, 00), (0, 01), (0, 10) ² , (4, 10)
B_3	100, 001 ² , 000 ²	100 ² , 010, 001 ² , 000
C_3	100 ² , 010	200, 001 ² , 000 ³
$A_2 \tilde{A}_1$	(10, 1) [±] , (10, 0) [±] , (00, 2), (00, 1) ²	(11, 0), (10, 2) [±] , (10, 1) [±] , (10, 0) [±] , (00, 2), (00, 1) ² , (00, 0) ²
$\tilde{A}_2 A_1$	(10, 1) [±] , (10, 0) [±] , (11, 0)	(20, 1) [±] , (20, 0) [±] , (11, 0), (00, 2), (00, 1) ² , (00, 0) ²

Next, the subgroups of E_6 in characteristic 3.

Subgroup of E_6 for $p = 3$	Factors on V_{\min}
$A_5 A_1$	(λ_4 , 0), (λ_1 , 1)
$A_2 A_2 A_2$	(10, 01, 00), (00, 10, 01), (01, 00, 10)
F_4	0001, 0000 ²
C_4	0100
$G_2 A_2$	(10, 10), (00, 02)
G_2 (2 classes)	20
D_5	$\lambda_1, \lambda_4, 0$
A_5	λ_1^2, λ_4
$A_4 A_1$	(1000, 1), (0001, 0), (0010, 0), (1, 0000)
$A_2 A_2 A_1$	(10, 01, 1), (01, 00, 1), (00, 10, 1), (01, 00, 0), (00, 10, 0)

Finally, the subgroups of E_7 in characteristics 7 and 11.

Subgroup of E_7 for $p = 7, 11$	Factors on V_{\min}
D_6A_1	$(\lambda_1, 1), (\lambda_5, 0)$
A_7	λ_1^\pm
A_5A_2	$(\lambda_1, 10)^\pm, (\lambda_3, 00)$
C_3G_2	$(001, 00), (100, 10)$
G_2A_1	$(01, 1), (10, 3)$
F_4A_1	$(0001, 1), (3, 0000)$
A_2	06^\pm
A_1A_1	$(6, 3), (4, 1), (2, 5)$
E_6	$\lambda_1^\pm, 0^2$
D_6	λ_1^2, λ_5
A_6	$\lambda_1^\pm, \lambda_2^\pm$
A_5A_1	$(\lambda_1, 10)^\pm, (\lambda_3, 00)$
A_4A_2	$(10, 1000)^\pm, (10, 0000)^\pm, (00, 0100)^\pm$
$A_3A_2A_1$	$(000, 10, 1)^\pm, (010, 00, 1), (100, 10, 0)^\pm, (100, 00, 0)^\pm$

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